NOTE ON CONNECTEDNESS IN TOPOLOGICAL RINGS

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We consider certain properties of topological rings with identity which can be deduced from connectedness.

The following two statements follow immediately from Kaplansky [2, Theorems 1 and 2].

1. A connected compact ring is a zero ring.
2. A connected locally compact ring with identity is a finite dimensional algebra over the reals. In this note we drop the assumption of local compactness. Most of the results are consequences of the following simple lemma.

Let $M$ be a topological module over a topological ring $A$, i.e. $M$ is a topological abelian group, a module over $A$, and $(a, m) \rightarrow am$ is jointly continuous.

**Lemma 1.** If $A$ is connected and $M_0$ is a submodule of $M$ then $AM_0$ is in the component of 0 in $M_0$.

**Proof.** Let $m_0 \in M_0$, then $a \rightarrow am_0$ is a continuous mapping of $A$ onto a connected subset of $M_0$. Since for $a = 0$ in $A$, $am_0 = 0$, this subset contains 0.

The following statements follow immediately from Lemma 1:

1. A unital module over a connected ring with identity has only connected submodules.
2. A connected ring with identity has only connected left (right) ideals.
3. The only discrete left (right) ideal in a connected ring with identity is the zero ideal.

We use the definition of covering space and simple connectedness given by Chevalley [1], and we prove the following theorem which is in direct analogy to Proposition 5, p. 53 of [1].

**Theorem 1.** If a ring $A$ admits a simply connected covering space $(S, f)$ then $S$ can be made into a ring so that $f$ is a ring homomorphism; furthermore if $A$ has an identity so does $S$.

**Proof.** Assume that $(S, f)$ covers $A$ and that $S$ is simply connected. Then the product $T = S \times S \times S \times S$ is also simply connected. Define the continuous mapping $\Omega: T \rightarrow A$ by $\Omega(a, b, c, d) = f(a)f(b) + f(c) - f(d)$. Let 0 be an element of $S$ contained in $f^{-1}(0)$, and let $\beta$ be a fixed element of $S$ not contained in $f^{-1}(0)$. The simple connected-
ness of $T$ then implies there is a unique "lifting" of $\Omega$ (a continuous mapping of $\Omega'$: $T \to S$ such that $f \circ \Omega' = \Omega$) such that $\Omega'(\beta, 0, 0, 0) = 0$. We define $-a$ in $S$ to be $\Omega'(\beta, 0, 0, a)$ and $a+b$ to be $\Omega'(\beta, 0, a, -b)$. These are continuous operations, $S$ becomes an abelian topological group under them, and $f$ is a group homomorphism of $S$ onto the additive group of $A$ (see Proposition 5, p. 53 of [1]).

We now define a (continuous) multiplication in $S$ by $ab = \Omega'(a, b, 0, 0)$. Note that $f(ab) = f(a)f(b)$.

We show first that $a0 = 0a = 0$ for all $a$ in $S$. The two continuous mappings $\theta: a \to 0$ and $\theta': a \to a0$ have the property that $f \circ \theta = f \circ \theta'$ and $\theta(\beta) = \theta'(\beta)$. Thus since $S$ is connected and $f$ is a covering map $\theta = \theta'$, $a0 = 0$. Also the two mappings $\theta$ and $\theta'': a \to a0$ obey $f \circ \theta = f \circ \theta''$ and agree on $a = 0$, thus $\theta = \theta''$ and $0a = 0$. We next show associativity.

The two continuous mappings $\Sigma: (a, b, c) \to a(bc)$ and $\Sigma'': (a, b, c) \to (ab)c$ from the connected space $S \times S \times S$ to $S$ have the property that $f \circ \Sigma = f \circ \Sigma''$ and $\Sigma$ and $\Sigma''$ agree on $(0, 0, 0)$. Thus $\Sigma = \Sigma''$ and $a(bc) = (ab)c$. To show left distributivity use the above argument applied to the two mappings $(a, b, c) \to a(b+c)$ and $(a, b, c) \to ab + ac$ and similarly for right distributivity.

If $A$ has an identity $e$ then the element $\beta$ may be chosen in $f^{-1}(e)$ and is an identity for $S$. This is because the two mappings $b \to \beta b$ and $b \to b$ have the same composition with $f$ and agree on $b = 0$. A similar argument shows $b\beta = b$. This completes the proof of the theorem.

As an immediate corollary we obtain

**Theorem 2.** A ring with identity which admits a simply connected covering space is already simply connected.

**Proof.** By Theorem 1, the ring admits a covering ring with identity and the covering map becomes a ring homomorphism. But the kernel of this is a discrete ideal and, by the property 3 listed above, must be zero. Thus the covering map is a homeomorphism.

**References**


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