ON THE INDEX OF A FIBERED MANIFOLD

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Introduction. Let $V$ be a real vector space of dimension $r$. Let $F(x, y) = \langle x, y \rangle$, $x, y \in V$, be a real-valued symmetric bilinear function. We can find a basis $e_i, 1 \leq i \leq r$, in $V$, such that

$$
F(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{r} x_i y_i
$$

where $x = \sum_{i=1}^{p} x_i e_i$ and $y = \sum_{i=p+1}^{r} y_i e_i$.

The number $p - q$ is called the index of $F$, to be denoted by $\tau(F)$. It depends only on $F$. If $F$ is nonsingular (i.e. $p + q = r$), then $\min(p, q)$ equals the maximal dimension of the linear subspaces of $V$ contained in the “cone” $F(x, x) = 0$.

Now let $M$ be a compact oriented manifold. The index of $M$ is defined to be zero, if the dimension of $M$ is not a multiple of 4. If $M$ has the dimension $4k$, consider the cohomology group $H^{2k}(M)$ with real coefficients. This is a real vector space, and the equation

$$
\langle x, y \rangle \xi = x \cup y, \quad x, y \in H^{2k}(M),
$$

where $\xi$ is the generator of $H^{4k}(M)$ defined by the given orientation of $M$, defines a real-valued symmetric bilinear form $\langle x, y \rangle$ over $H^{2k}(M)$. Its index is called the index of $M$, to be denoted by $\tau(M)$. Reversal of the orientation of $M$ changes the sign of the index. The form $\langle x, y \rangle$ defined by (2) is nonsingular, since, by Poincaré's duality theorem, the equation $x \cup y = 0$ for all $x \in H^{2k}(M)$ implies $y = 0$.

The main purpose of this paper is to prove the theorem:

**Theorem.** Let $E \rightarrow B$ be a fiber bundle, with the typical fiber $F$, such that the following conditions are satisfied:

1. $E$, $B$, $F$ are compact connected oriented manifolds;
2. The fundamental group $\pi_1(B)$ acts trivially on the cohomology ring $H^*(F)$ of $F$.

Then, if $E$, $B$, $F$ are oriented coherently, so that the orientation of $E$ is induced by those of $F$ and $B$, the index of $E$ is the product of the indices of $F$ and $B$, that is,

$$
\tau(E) = \tau(F) \tau(B).
$$
Remark. We do not know whether condition (2) and the connectedness hypothesis of condition (1) are necessary. For instance, let $E$ be an $n$-sheeted covering of $B$ (the spaces $B$ and $E$ still being compact oriented manifolds); is it true that $\tau(E) = n\tau(B)$? We know the answer to be positive only when $B$ possesses a differentiable structure: in that case, according to a theorem of one of us, $\tau(B)$ (resp. $\tau(E)$) is equal to the Pontrjagin number $L(B)$ (resp. $L(E)$) and it is clear that $L(E) = n \cdot L(B)$.

1. Algebraic properties of the index of a matrix. Let $e_i$, $1 \leq i \leq r$, be a base in $V$. A real-valued symmetric bilinear function $\langle x, y \rangle$ defines a real-valued symmetric matrix $C = (c_{ij})$, $c_{ij} = \langle e_i, e_j \rangle$, $1 \leq i, j \leq r$, and is determined by it. The index of the bilinear function is equal to the index $\tau(C)$ of $C$, if we define the latter to be the excess of the number of positive eigenvalues over the number of negative eigenvalues of $C$, each counted with its proper multiplicity. We have the following properties of the index of a real symmetric matrix:

For a nonsingular $(r \times r)$-matrix $T$ we have

$$\tau(C) = \tau(T^T C T).$$

Here, as always, we denote by $T^T$ the transpose of $T$. For nonsingular square matrices $A$, $L$ (with $A$ symmetric) we have

$$\tau\begin{pmatrix} 0 & 0 & L \\ 0 & A & 0 \\ tL & 0 & 0 \end{pmatrix} = \tau\begin{pmatrix} 0 & L \\ tL & 0 \end{pmatrix} + \tau(A) = \tau(A).$$

Here and always we make use of the convention that the index of the empty matrix is zero.

To prove (4) it is enough to show that

$$\tau\begin{pmatrix} 0 & L \\ tL & 0 \end{pmatrix} = 0.$$ 

In this case, $r$ is even. Put $r = 2\mu$. Obviously, the cone $F(x, x) = 0$ of the symmetric bilinear function $F(x, y) = 0$ belonging to the matrix

$$\begin{pmatrix} 0 & L \\ tL & 0 \end{pmatrix}$$

contains a linear space of dimension $\mu$. Thus $\min(p, q) \geq \mu$. On the other hand, $p + q = 2\mu$. Therefore, $p = q$ and $\tau = 0$.

Lemma 1. Let $C$ be a real, symmetric, nonsingular matrix of the form
where \( L_0, \ldots, L_m \) are square matrices (empty matrices are admitted) and where \( L_i \) is the transpose of \( L_{m-i} \). Then

\[
\tau(C) = \tau\begin{pmatrix}
0 & L_0 \\
& \\
& \\
L_m & 0
\end{pmatrix} = \begin{cases}
0, & \text{if } m \text{ is odd}, \\
\tau(L_n), & \text{if } m = 2n.
\end{cases}
\]

Proof. We put

\[
C_\lambda = \begin{pmatrix}
0 & L_0 \\
& \\
& \\
L_m & \lambda
\end{pmatrix}, \quad 0 \leq \lambda \leq 1.
\]

Since \( \det(C_\lambda) = \pm \prod_{i=0}^{m} \det(L_i) \neq 0 \), the index \( \tau(C_\lambda) \) is obviously independent of \( \lambda \), so that \( \tau(C) = \tau(C_1) = \tau(C_0) \). By (4) we have \( \tau(C_0) = 0 \) resp. \( \tau(C_0) = \tau(L_n) \), q.e.d.

Lemma 2. Let \( A \) and \( B \) be two square matrices, which are either both symmetric or both skew-symmetric. Then their tensor product \( A \otimes B \) is symmetric, and

\[
\tau(A \otimes B) = \tau(A)\tau(B) \text{ or } 0,
\]

according as both \( A \) and \( B \) are symmetric or skew-symmetric.

Suppose first that \( A \) and \( B \) are both symmetric. Let \( \alpha_i > 0, \alpha_j < 0, 1 \leq i \leq p, p+1 \leq j \leq p+q \), be the nonzero eigenvalues of \( A \) and \( \beta_k > 0, \beta_{k'} < 0, 1 \leq k \leq p', p'+1 \leq l \leq p'+q' \) be the nonzero eigenvalues of \( B \). Then the nonzero eigenvalues of \( A \otimes B \) are \( \alpha_u\beta_k, 1 \leq u \leq p+q, 1 \leq t \leq p'+q' \). It follows that

\[
\tau(A \otimes B) = pp' + qq' - pq' - p'q = \tau(A)\tau(B).
\]

Now let \( A \) and \( B \) be both skew-symmetric. By applying (3) to the matrix \( C = A \otimes B \) we can suppose that \( A \) and \( B \) are both of the form

\[
\begin{pmatrix}
A_1 & 0 \\
& \\
& \\
& A_n \\
0 & 0
\end{pmatrix}
\]
where each $A_i$ is a $2 \times 2$ block:

$$A_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J.$$  

Since

$$\tau \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \tau \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = 0,$$

we have $\tau(A \otimes B) = 0$.

2. Poincaré rings. We consider a graded ring $A$ with the following properties:

(1) In the direct sum decomposition

$$A = \sum_{0 \leq r < \infty} A^r$$

of $A$ into the subgroups of its homogeneous elements, each $A^r$ is a real vector space of finite dimension. There exists an $n$ with $A^r = 0$ for $r > n$ and with $\dim A^n = 1$.

(2) If $x \in A^i$, $y \in A^j$ then $xy \in A^{i+j}$ and

$$xy = (-1)^{ij}yx.$$

Let $\xi \neq 0$ be a base element of $A^n$. Relative to $\xi$ we define a bilinear pairing $\langle x, y \rangle$ of $A^r$ and $A^{n-r}$ into the real field by the equation

$$\langle x, y \rangle \xi = xy, \quad x \in A^r, \ y \in A^{n-r}.$$  

Let $i_{n-r}$ be the linear mapping of $A^{n-r}$ into $(A^r)^*$, the dual vector space of $A^r$, which assigns to $y \in A^{n-r}$ the linear function $\langle x, y \rangle$ on $A^r$ ($x \in A^r$).

A graded ring $A$ is called a Poincaré ring if it satisfies (1), (2) and has moreover the following property:

(3) The mapping $i_{n-r}$ is a bijection of $A^{n-r}$ onto $(A^r)^*$.

A consequence of (3) is

$$\dim A^r = \dim A^{n-r}, \quad 0 \leq r \leq n.$$  

The cohomology ring of a compact orientable manifold is a Poincaré ring.

A differentiation in a Poincaré ring $A$ is a linear endomorphism $d: A \to A$, satisfying the following conditions:

(a) $dA^r \subset A^{r+1}$;

(b) $dd = 0$;

(c) $d(xy) = (dx)y + (-1)^r x(dy)$, if $x \in A^r$;

(d) $dA^{n-1} = 0$.  

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As is well known, such a differentiation defines a derived ring \( A' = d^{-1}(0)/dA \). If we put \( A'^r = d^{-1}(0) \cap A^r/dA^{r-1} \), we have the direct sum decomposition

\[ A' = \sum_{0 \leq r \leq n} A'^r, \]

and \( A' \) is a graded ring. It is easy to verify that, if \( x' \in A'^i, y' \in A'^j \), then \( x'y' \in A'^{i+j} \), and

\[ x'y' = (-1)^{ij} y'x'. \]

From the property (\( \delta \)) of \( d \) we have \( \dim A'^n = 1 \). Thus \( A' \) satisfies (1) and (2) with the same maximal degree \( n \) as \( A \). We denote the residue class of \( \xi \) in \( A'^n \) by \( \xi' \). Relative to \( \xi' \) we have the linear mapping

\[ \iota_{n-r}: A'^{n-r} \to (A'^r)^*. \]

**Lemma 3.** The derived ring of a Poincaré ring with differentiation is a Poincaré ring, i.e. \( \iota_{n-r} \) is bijective.

It remains to prove that \( A' \) has the property (3) in the definition of a Poincaré ring. Let \( x \in A^r, y \in A^{n-r-1} \). By property (\( \delta \)) of \( d \), we have

\[ 0 = d(xy) = (dx)y + (-1)^rx(dy). \]

This gives

\[ \langle dx, y \rangle = (-1)^{r-1}\langle x, dy \rangle, \]

a relation which is independent of the choice of \( \xi \). This relation is equivalent to saying that the following diagram is commutative:

\[ \begin{array}{ccc}
A^{n-r-1} & \xrightarrow{d} & A^{n-r} & \xrightarrow{d} & A^{n-r+1} \\
\downarrow \iota_{n-r-1} & & \downarrow \iota_{n-r} & & \downarrow \iota_{n-r+1} \\
(A^{r+1})^{*} & \xrightarrow{(-1)^{r-1}(td)} & (A^{r})^{*} & \xrightarrow{(-1)^{r}(td)} & (A^{r-1})^{*}
\end{array} \]

where \((A^{r})^{*}\) is the dual space of \( A^{r} \), and \( 'd \) is the dual homomorphism of \( d \). We have the canonical isomorphism

\[ (A'^r)^* \cong 'd^{-1}(0) \cap (A^{r})^{*}/'d(A^{r+1})^{*}. \]

The above diagram shows that \( \iota_{n-r} \) induces an isomorphism, namely \( \iota_{n-r}^{*} \), of \( A'^{n-r} \) onto \((A'^r)^*\). It follows that \( A'^r \) and \( A'^{n-r} \) are dually paired into the real field relative to the element \( \xi' \in A'^n \), which is the residue class of \( \xi \).
In analogy with the index of an oriented manifold we can define the index $\tau_\xi(A)$ of our Poincaré ring $A$ relative to $\xi$. It is to be zero, if $n \equiv 0$, mod 4. If $n = 4k$, $\tau_\xi(A)$ is to be the index of the bilinear function $\langle x, y \rangle$, $x, y \in A^{2k}$. Obviously, $\tau_\xi(A) = \tau_{\xi'}(A')$, if $\xi'$ is a positive multiple of $\xi$.

**Lemma 4.** In a Poincaré ring $A$ let $\xi \neq 0$ be a base of $A^n$, and let $\xi' \in A'^n$ be the residue class which contains $\xi$. Then $\tau_{\xi'}(A') = \tau_\xi(A)$.

It is only necessary to prove the lemma for the case $n = 4k$. Let $Z^{2k} = d^{-1}(0) \cap A^{2k}, B^{2k} = dA^{2k-1}$, and let $a, b, c$ be the respective dimensions of $A^{2k}, B^{2k}, Z^{2k}$. It follows immediately from (8) that each of the two spaces $B^{2k}$ and $Z^{2k}$ is the orthogonal of the other with respect to the symmetric form $\langle x, y \rangle$ of $A^{2k}$, whence $a = b + c$. We have $B^{2k} \subset Z^{2k} \subset A^{2k}$. If $e_i$ is a base of $A^{2k}$ such that $e_i \in B^{2k}$ for $1 \leq i \leq b$ and $e_i \in Z^{2k}$ for $b + 1 \leq i \leq c$, the matrix $((e_i, e_j))$ has then the form

$$
\begin{pmatrix}
0 & 0 & L \\
0 & Q & * \\
L & * & *
\end{pmatrix},
$$

where $L$ and $Q$ are square nonsingular matrices, of orders $b$ and $c - b$ respectively. Its index is $\tau_\xi(A)$, while $\tau(Q)$ is $\tau_{\xi'}(A')$. By Lemma 1, we get therefore $\tau_{\xi'}(A') = \tau_\xi(A)$, as contended.

**3. Proof of the theorem.** It suffices to prove the theorem (see Introduction) for the case dim $E = 4k$, which we suppose from now on. We consider the cohomology spectral sequence $E^r_{p,q}, 2 \leq r \leq \infty$, of the bundle $E \to B$, with the real field as the coefficient field. Let

$$
E^s_r = \sum_{p+q=s} E^p_q, \quad E_r = \sum_{0 \leq s \leq r} E^s_r, \quad 2 \leq r \leq \infty.
$$

Each $E_r$ is a graded ring, satisfying $E^p_r E^s_r \subset E^{p+s}_r$ and also $E^p_r E^{p',q'}_r \subset E^{p+p',q+q'}_r$. It has a differentiation $d_r$, such that $E_{r+1}$ is the derived ring of $E_r$. In our case $d_r$ is trivial for sufficiently large $r$ and $E_\infty$, or $E_r$ for $r$ sufficiently large, is the graded ring belonging to a certain filtration of the cohomology ring of the manifold $E$. The term $E_2$ of the spectral sequence is by hypothesis (2) of our theorem isomorphic to $H^*(B, H^*(F)) = H^p(B) \otimes H^q(F)$, such that

$$
E^p_q \cong H^p(B, H^q(F)) \cong H^p(B) \otimes H^q(F).
$$

If we identify $E^p_q$ with $H^p(B) \otimes H^q(F)$ under this isomorphism, the multiplication in $E_2$ is given by
\[(b \otimes f)(b' \otimes f') = (-1)^{\nu \cdot \nu'}(b \cup b') \otimes (f \cup f'),\]
\[b \in H^\nu(B), \quad b' \in H^\nu'(B), \quad f \in H^\nu(F), \quad f' \in H^\nu'(F).\]

Let \(m = \dim F\), so that \(\dim B = 4k - m\). Since \(B\) and \(F\) are manifolds, \(E_2\) is a Poincaré ring with respect to the grading

\[E_2 = \sum_{0 \leq s < \infty} E_2^s \quad (E_2^s = 0 \text{ for } s > 4k, \ E_2^{4k} = E_2^{4k-m,m}).\]

The ring \(E_2\) is isomorphic to the cohomology ring of \(B \times F\).

The orientations of \(B\), \(F\) define a generator \(\xi_b = \xi_B \otimes \xi_F\) of \(E_2^{4k}\). Here \(\xi_B\) (resp. \(\xi_F\)) denotes the generator of \(H^{4k-m}(B)\) (resp. \(H^m(F)\)) belonging to the orientation of \(B\) (resp. \(F\)). We wish to prove that

\[\tau_{\xi_b}(E_2) = \tau(B) \cdot \tau(F).\]

We have

\[(9) \quad E_2^{2k} = E_2^{2k,0} + E_2^{2k-1,1} + \cdots + E_2^{2k-m,m}.\]

Here some of the \(E_2^{p,q}\) might vanish, in particular \(E_2^{p,q} = 0\) if \(p < 0\). Clearly, for \(x \in E_2^{2k-a,a}\) and \(y \in E_2^{2k-d,d'}\) we have \(xy = 0\) unless

\[q + q' = m.\]

By Poincaré duality in \(B\) and \(F\), we have

\[\dim E_2^{2k-a,a} = \dim E_2^{2k-m+a,m-a}.\]

Therefore, the symmetric matrix, which defines the bilinear symmetric function over \(E_2^{2k}\), is, when written in blocks relative to the direct sum decomposition (9), of the form

\[
\begin{bmatrix}
  0 & L_0 \\
  \vdots & \ddots \\
  L_m & \cdots & 0
\end{bmatrix}
\]

where the \(L_i\) are nonsingular square matrices, such that \(L_i\) is the transpose of \(L_{m-i}\). By Lemma 1 we obtain

\[\tau_{\xi_b}(E_2) = 0 \text{ if } m \text{ is odd,} \quad \tau_{\xi_b}(E_2) = \tau(L_{m/2}) \text{ if } m \text{ is even.}\]

In the first case the equation \(\tau_{\xi_b}(E_2) = \tau(B) \tau(F)\) is proved, since \(\tau_{\xi_b}(E_2) = \tau(F) = 0\). In the latter case we have

\[
E_2^{2k-m/2,m/2} = H^{2k-m/2}(B) \otimes H^{m/2}(F),
\]
and it is clear that up to the sign \((-1)^{m/2}\) the matrix \(L_{m/2}\) is the tensor product of the two matrices defining the bilinear forms of \(B\) and \(F\). If \(m/2\) is odd, both matrices in this tensor product are skew-symmetric, and we have, by Lemma 2, \(\tau(L_{m/2}) = 0\); on the other hand we have \(\tau(B)\tau(F) = 0\), since \(\dim F \equiv 0 \pmod{4}\) and thus by definition \(\tau(F) = 0\). If \(m/2\) is even, that is, if \(m \equiv 0 \pmod{4}\), both matrices are symmetric, and Lemma 2 gives: \(\tau(L_{m/2}) = \tau(B)\tau(F)\). Combining all cases, we get the formula

\[
\tau(E_2) = \tau(B)\tau(F)
\]
in full generality.

The differentiation \(d_2\) of \(E_2\) satisfies the conditions of a differentiation in a Poincaré ring given in §2. In fact, \(\dim E_4^{4k} = 1\), since \(E\) is a manifold of dimension \(4k\). Therefore, \(\dim E_r^{4k} = 1\) for \(2 \leq r\). Thus \(d_2\) annihilates \(E_2^{4k-1}\); more generally \(d_r\) annihilates \(E_r^{4k-1}\). It follows by Lemma 3 that \(E_3\) is a Poincaré ring. It has \(d_3\) as differentiation and therefore \(E_4\) is a Poincaré ring etc. Finally, \(E_\infty\) is a Poincaré ring. By Lemma 4 and (10) we get

\[
\tau(B)\tau(F) = \tau(E_2) = \tau(E_3) = \cdots = \tau(E_\infty),
\]
where \(\xi_r\) (resp. \(\xi_\infty\)) is the image of \(\xi_2\) in \(E_r\) (resp. \(E_\infty\)).

It remains to prove that \(\tau_{E_\infty}(E_\infty) = \tau(E)\). The cohomology ring \(H^*(E)\) is filtered:

\[
H^*(E) = D^0 \supset D^1 \supset \cdots \supset D^p \supset D^{p+1} \supset \cdots, \quad \cap D^p = 0,
\]

\[
D^{p,q} = D^p \cap H^{p+q}(E),
\]

\[
D^{p,q} \cdot D^{p',q'} \subset D^{p+p',q+q'}.
\]

We have the filtration

\[
H^r(E) = D^{0,r} \supset D^{1,r-1} \supset \cdots \supset D^{r,0} \supset D^{r+1,-1} = 0
\]
and the canonical isomorphism

\[
D^{p,q}/D^{p+1,q-1} \cong E_\infty^{p,q}.
\]

The ring structure of \(E_\infty\) is induced by that of \(H^*(E)\) by the canonical homomorphisms \(D^{p,q} \to E_\infty^{p,q}\) (see (12) and (11)). Since \(E_\infty^{4k} = E_\infty^{4k-m,m}\), (where \(m = \dim F\)), we have

\[
H^{4k}(E) = D^{4k-m,m} \cong E_\infty^{4k-m,m}
\]
and

\[
D^{4k-i,i} = 0 \quad \text{for } i < m.
\]
Earlier we have chosen a generator $\xi_{\infty} \in E^{k}_{0}$. Under the canonical isomorphism (13) $\xi_{\infty}$ goes over in the generator $\xi_{E}$ of $H^{2k}(E)$ belonging to the orientation of $E$ generated by the given orientations of $B$ and $F$ in this order.\textsuperscript{2} We now consider the bilinear symmetric function $\langle x, y \rangle$ over $H^{2k}(E)$ relative to $\xi_{E}$. Choose a direct sum decomposition of $H^{2k}(E)$ in linear subspaces,

$$H^{2k}(E) = V_{0} + V_{1} + V_{2} + \cdots + V_{m}$$

such that

$$\sum_{i=0}^{q} V_{i} = D^{2k-q} \quad (0 \leq q \leq m).$$

Here we use that $D_{2k-s} = D_{2k-m,m} \text{ for } s > m$. By (11) and (14) we have

$$\langle x, y \rangle = 0 \quad \text{for } x \in V_{i}, y \in V_{j} \text{ and } i + j < m,$$

and moreover by (13)

$$\langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle, \quad \text{for } x \in V_{i}, y \in V_{j} \text{ and } i + j = m,$$

where $\bar{x}$ (resp. $\bar{y}$) denotes the image (see (12)) of $x$ (resp. $y$) in $E^{2k-i}_{0}$ (resp. $E^{2k-j}_{0}$) and where on the right side of this equation stands the symmetric bilinear form over $E^{2k}_{0}$ relative to $\xi_{\infty}$. Since $\langle \bar{x}, \bar{y} \rangle = 0$ for $\bar{x} \in E^{2k-q}_{0}, \bar{y} \in E^{2k-q'}_{0}$, unless $q + q' = m$, and since $E_{\infty}$ is a Poincaré algebra, we can conclude

$$\dim E^{2k-q}_{\infty} = \dim E^{2k-m+q,m-q}_{\infty}.$$

The preceding remarks, in particular (16), (17), (18), imply: The matrix of the symmetric bilinear function over $H^{2k}(E)$ relative to $\xi_{E}$ can be written in blocks with respect to the direct sum decomposition (15) in the form

$$\begin{bmatrix}
0 & L_{0} \\
& L_{1} \\
& \ddots \\
& L_{m} & *
\end{bmatrix}$$

\textsuperscript{2} This is easy to see when $E$ is a trivial bundle, in which case it is almost the definition of the orientation of a product of manifolds. The general case can be reduced to this one by comparing the spectral sequence of $E$ to that of the bundle induced by $E$ on an open cell of the base, the cohomology being taken with compact carriers.
where the $L_i$ are nonsingular square matrices and where $L_i$ is the transpose of $L_{m-i}$. Moreover,

$$
\begin{pmatrix}
L_0 \\
\vdots \\
L_m \\
0
\end{pmatrix}
$$

is the matrix of the symmetric bilinear function over $E^{2k}_\infty$ relative to $\xi$. By Lemma 1 we have $\tau(E) = \tau_{E_\infty}(E_\infty)$. This concludes the proof of our theorem.

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THE PERIPHERAL CHARACTER OF CENTRAL ELEMENTS OF A LATTICE

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A lattice being a Hausdorff space together with a pair of continuous lattice operations ($\land$ and $\lor$) the content of this note is best exhibited by quoting a corollary to our theorem: If a compact connected lattice is (topologically) situated in Euclidean $n$-space then its center is contained in its boundary. Thus, far from being "centrally located," the central elements are "peripheral."

The above is a consequence (see [3, p. 273]) of the

Theorem. If $L$ is a compact connected lattice, if $R$ is an $(r, G)$-rim [3] for $L$ and if (i) a is central [1, p. 27] or if (ii) $L$ is modular and a is complemented then $a \in R$.

Proof. The procedure is to introduce an appropriate multiplication into $L$ so that $L$ is a semigroup, to show that $L$ is not simple (in the semigroup sense [3]) and that $a$ is a left unit. Since $L$ is compact it has a zero and unit, 0 and 1, as is well-known. Indeed, the set $\cap \{x \lor L \mid x \in L\}$ is easily seen to consist of exactly one element, namely 1. If $a = 1$ then the hypotheses of Theorem 1 of [3] are fulfilled using the multiplication $(x, y) \to x \land y$ so that 1 being a unit for the multiplication, $1 \in R$. If $a \neq 1$ let $x \cdot y = (a' \land x) \lor y$, $a'$ being a

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