1. Introduction. A set $S$ in Euclidean $E_n$—or, equivalently, on the $n$-sphere $S_n$—is called \textit{inversely convex} if there is an inversion on an $(n-1)$-sphere in the space which transforms $S$ into a convex set. A simple closed $(n-1)$-surface is inversely convex if one of the two regions bounded by it is inversely convex.

The group generated by the inversions can be characterized as follows [1, p. 102].

\textbf{Lemma 1.} Let $T$ be a product of inversions of $E_n$. Then $T$ is either a similarity or the product of an isometry and an inversion. The product of two concentric inversions is a similarity.

\textbf{Proof.} If $T$ keeps the point at infinity fixed then it transforms the planes (spheres through infinity) into planes and is therefore affine. Since $T$ is also conformal it is therefore a similarity.

Now let $T$ transform infinity into the finite point $p$ and let $I_p$ be an inversion on a sphere with center $p$ (inversion at $p$ for short). Then $I_pT$ keeps infinity fixed and is a similarity; or, by a suitable choice of radius for the sphere of $I_p$, an isometry $U$. Thus $T = I_p U$. The last statement of the lemma follows from the fact that such a product leaves infinity fixed.

As a consequence of Lemma 1 we see that inversive convexity is equivalent to convexity under an element of the group $G$ generated by inversions. In the (complex) plane (Riemann sphere) $G$ is the group of conformal and anticonformal linear fractional transformations. For $n > 2$ the group $G$ is the group of all conformal transformations [1, p. 100] and inversive convexity is therefore equivalent to \textit{conformal convexity}. As another consequence of Lemma 1 we may now speak of \textit{inversive convexity at $p$}, meaning convexity under inversion on the spheres with center $p$.

In §2 we give characterizations of inversive convexity of sets and curves in $E_2(S_2)$. Moreover, we characterize the centers of inversion which transform a given set into a convex set. In §3 new characterizations are given which are valid also for $n > 2$ to yield a characterization of conformal convexity in $E_n(S_n)$.

2. Inversive convexity in the plane. A first simple characterization
of inversive convexity at $p$, valid regardless of dimension, is given by the following.

**Theorem 1.** The set $S$ is inversively convex at $p$ if and only if for every two points $a, b \in S$ the arc $ab$ of the circle determined by $a, b, p$ which does not contain $p$, is contained in $S$. (If $p = a$ this reduces to the statement that $p, b$ are joined by some circular arc in $S$.)

The proof is obvious if we remember that the arc $ab$ in question goes into the segment $a'b'$ of $S'$, where $x'$ indicates the image of $x$ upon inversion at $p$.

From Theorem 1 we see that inversive convexity implies a certain kind of circle convexity. If $S$ is inversively convex at two points $p, q$ then we get a "lune convexity" from Theorem 1.

In order to obtain a more useful characterization we now pass to the question of inversive convexity of curves. A curve $C$ in $E_2 (S_2)$ is called convex if it lies in the boundary of its convex hull. A convex curve is thus either a simple arc (possibly with one endpoint at infinity) or a simple closed curve (possibly through infinity).

**Lemma 2.** An oriented simple curve $C$ in the plane is convex if and only if all triangles $\triangle abc$ with vertices appearing in that order on $C$ have the same orientation. (Collinear triples may be excluded or permitted according as we wish to define strict convexity or not.)

**Proof.** The interior of $\triangle abc$ lies in the convex hull $H(C)$ of $C$. Thus the orientation given to the boundary of $H(C)$ by that of $C$ is the same as that of $\triangle abc$.

Conversely if $C$ is not convex then the boundary of $H(C)$ contains at least three points $a, b, c$ of $C$ so that $\triangle abc$ contains in its interior a point $d \in C$. The four triangles determined by $a, b, c, d$ can then not have the same orientation.

Now, on the simple, oriented curve $C$, each triple of points $a, b, c$ determines a circle $K_{abc}$, whose orientation is given by the order given to $a, b, c$ by the orientation of $C$. We label the open (closed) side of $K_{abc}$ corresponding to one orientation of the plane by $S^+ (a, b, c)$ (respectively $S^- (a, b, c)$) and the open (closed) side corresponding to the opposite orientation by $S^- (a, b, c)$ (respectively $S^+ (a, b, c)$). We now define the intersections

\[
S^+ = \bigcap_{a, b, c \in C} S^+ (a, b, c); \quad S^- = \bigcap_{a, b, c \in C} S^- (a, b, c),
\]

\[
S^- = \bigcap_{a, b, c \in C} S^- (a, b, c); \quad S^- = \bigcap_{a, b, c \in C} S^- (a, b, c).
\]
Since inversion at a point \( p \) preserves the orientation of every circle containing \( p \) in its interior and reverses the orientation of every circle containing \( p \) in its exterior, Lemma 2 yields the following.

**Theorem 2.** A simple, oriented curve \( C \) becomes convex upon inversion at \( p \) if and only if \( p \in S_+^{k} \cup S_-^{k} \), and becomes strictly convex if and only if \( p \in S_+^{k} \cup S_-^{k} \).

**Proof.** From our construction we see that a point \( p \) lies in \( S_+^{k} \) or \( S_-^{k} \) if and only if it lies in the closed interior of all \( K_{abc} \) with one orientation of \( \Delta abc \) and in the closed exterior of all \( K_{abc} \) with the opposite orientation of \( \Delta abc \). Thus by the remark preceding the theorem, upon inversion at \( p \) the image triples which are not collinear will be triangles which all have the same orientation. Thus according to Lemma 2 the image \( C' \) is convex. If \( p \in S_+^{k} \cup S_-^{k} \) then no triple \( a, b, c \) lies on a circle through \( p \). Thus collinearity of \( a', b', c' \) is excluded and \( C' \) is strictly convex.

Conversely if \( p \) lies on opposite sides of \( K_{abc} \) and \( K_{def} \) then the images \( \Delta a'b'c' \) and \( \Delta d'ef' \) upon inversion at \( p \) have opposite orientation and \( C' \) is not convex. Finally if \( p \in (S_+^{k} - S_-^{k}) \cup (S_-^{k} - S_+^{k}) \) then \( p \) lies on one of the circles \( K_{abc} \). Thus the image \( C' \) contains collinear \( a', b', c' \) and is not strictly convex.

**Corollary.** Let \( S \) be the exterior of a closed convex curve \( C \) (possibly through infinity) and \( P \) the set of points at which \( S \) is inversively convex, then \( P \) is a convex (possibly empty) closed subset of the closed interior of \( C \).

Instead of convexity we now consider the concept of local convexity of a curve.

**Definition.** An oriented curve \( C \) is **locally convex** if every point \( x \) on \( C \) is contained in an open convex arc \( C_x \) of \( C \) so that the convex hulls \( H(C_x) \) lie on the same side of \( C \). A curve \( C \) is **inversely locally convex** if its image upon inversion on some circle is locally convex.

If the \( C_x \) are strictly convex then the condition on \( H(C_x) \) becomes superfluous.

A locally convex curve need not be convex and need not even be simple. However a simple closed curve (possibly through infinity) which is locally convex is convex. Thus for simple closed curves the characterization to be given of inverse local convexity is a characterization of inverse convexity.

**Definition.** A **limit circle** of a curve \( C \) at a point \( x \in C \) is the limit of a sequence of circles \( K_{abc} \) with \( a, b, c \in C \) as the arc \( a, b, c \) shrinks to the point \( x \). If \( C \) is oriented then the approximating circles are oriented and we obtain **oriented limit circles** (possibly of zero radius).
Lemma 3. An oriented curve $C$ is locally convex if and only if all its limit circles have the same orientation.

Proof. Each point $x \in C$ is contained in a convex subarc $C_x \subseteq C$. Hence by Lemma 2 all $K_{abc}$ with $a, b, c \in C_x$ have the same orientation, which is also the orientation of any limit circle at $x$. Now let $x, y$ be two points on $C$. By hypothesis $C_x$ and $C_y$ have the same orientation relative to their convex hulls. Hence the limit circles at $x$ and $y$ have the same orientation.

Assume now that $C$ is not locally convex. Then either (i) there exists a point $x \in C$ so that every open subarc $C_x \subseteq C$ which contains $x$, contains points $a, b, c$ so that $x$ is an interior point of $\Delta abc$; or (ii) $C$ contains a straight line segment $xy$ so that $H(C_x), H(C_y)$ are on opposite sides of $xy$. In case (i) not all the circles determined by three of the four points $a, b, c, x$ can have the same orientation. If we let $C_x$ shrink to $x$ we thus obtain limit circles at $x$ with opposite orientations. In case (ii) the limit circles of $C_x$ and of $C_y$ are on opposite sides of $C$.

Theorem 3. Let $C$ be an oriented curve. Let $\mathcal{S}_x^+$ be the set of points which lie on one closed side of all the limit circles of $C$ and $\mathcal{S}_x^-$ the set of points lying on the other closed side of all the limit circles of $C$. Then $C$ is locally convex upon inversion at $p$ if and only if $p \in \mathcal{S}_x^+ \cup \mathcal{S}_x^-$. The derivation of Theorem 3 from Lemma 3 is entirely analogous to that of Theorem 2 from Lemma 2 and is therefore omitted.

If $C$ is differentiable the limit circles are the circles of curvature of $C$, and Theorem 3 could then be stated in terms of circles of curvature. Another interesting special case is the following.

Corollary. If $C$ is a closed convex curve (possibly through infinity), then the intersection $S$ of the closed interiors of the circles $K_{abc}$ ($a, b, c \in C$) is also the intersection of the closed interiors of the limit circles of $C$. (The "interior" of a circle of infinite radius is that half plane which contains $C$.) The set $S$ is the set of points at which the exterior of $C$ is inversively convex.

If the curvature of $C$ is bounded away from zero—or, more generally, if the limit circles of $C$ have bounded radius, then the interior of $C$ is inversively convex at every point sufficiently far from $C$.

3. Conformal convexity. Instead of extending the method of §2 to higher dimensions—which is possible in part but cumbersome—we introduce a new method which gives new insight also in the two-dimensional case.

Definition. A sphere of support of a set $S$ at a point $x$ in the closure
S of $S$ is a sphere $K$ such that $S$ lies entirely on one closed side of $K$. An extremal sphere of support of $S$ at $x$ is a sphere of support of $S$ at $x$ such that the side containing $S$ (we shall call it the positive side) is minimal.

An extremal sphere of support may be a plane or degenerate to a single point. There may be several extremal spheres of support at a point.

Now we associate to each boundary point $x$ of $S$ the set $K_x$ consisting of the union of the closed negative sides of the extremal spheres of support of $S$ at $x$. Finally we let $K_S$ be the intersection of the $K_x$ as $x$ runs through all boundary points of $S$.

Lemma 4. A set $S$ with interior points is inversely convex at $p$ if and only if there is a supporting sphere through $p$ at every boundary point of $S$.

Proof. The image $S'$ of $S$ upon inversion at $p$ is convex if and only if there is a plane of support to $S'$ at every boundary point of $S'$. Inverting again we transform the planes into spheres of support of $S$ through $p$.

Theorem 4. A set $S$ with interior points is inversely convex at $p$ if and only if $p \in K_S$. Thus inversive convexity of $S$ is equivalent to the nonemptiness of $K_S$.

Proof. If $S$ is inversely convex at $p$ then according to Lemma 4 there is a sphere of support to $S$ through $p$ at every boundary point $x$ of $S$. This sphere is contained in the closed negative side of some extremal sphere of support to $S$ at $x$ and hence $p \in K_x$ for every boundary point $x$, that is $p \in K_S$.

Conversely, assume $p \in K_S$, then for every boundary point $x$ of $S$ there is an extremal sphere of support $K$ which contains $p$ in its closed negative side. The sphere through $p$ and tangent to $K$ at $x$ is therefore in the closed negative side of $K$ and is a sphere of support to $S$ at $x$ (in case $x = p$ this choice is not unique). Thus, according to Lemma 4, $S$ is inversively convex at $p$.

A combination of Theorem 4 with the Corollary to Theorem 3 yields some information about convex curves, (the analogous theorem also holds in higher dimensions).

Corollary. For a closed convex twice differentiable curve $C$ in the plane the following sets are identical:

(i) The intersection of the closed interiors of the circles of curvature of $C$, and the intersection of the closed interiors of the maximal inscribed circles to $C$. 

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(ii) The intersection of the closed exteriors of the circles of curvature of $C$, and the intersection of the closed exteriors of the minimal circumscribed circles to $C$.

Reference


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HOMOTOPY GROUPS OF ONE-DIMENSIONAL SPACES

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In this paper we prove the following theorem:

If $S$ is a one-dimensional separable metric space, then $\pi_k(S) = 0$ for all $k > 1$.

Actually it is proved that a much broader class of spaces than spheres have the property that mappings of these spaces into one-dimensional spaces are homotopic to constant maps. This class of spaces includes, for example, projective spaces and Lens spaces.

Lemma 1. Let $X$ be a compact metric space whose one-dimensional integral singular homology group is a torsion group. Then for any finite covering $G$ of order one by arcwise-connected open sets, $G$ does not contain a simple loop.

Proof. By a simple loop we mean a simple chain such that the first and last sets are the same. Let $K$ be the nerve of $G$. Since $K$ is one-dimensional, a simple loop in $G$ implies a nonbounding one cycle in $K$. Hence it suffices to show that $H_1(K) = 0$.

Let $\phi: X \to K$ be a canonical map. For each vertex $v$ in $K$ we choose a point $\psi(v)$ in the element of $G$ corresponding to $v$. For each edge $\sigma$ with vertices $v_1$ and $v_2$ we extend $\psi$ on $\{v_1, v_2\}$ to a mapping of $\sigma$ into the union of the two elements of $G$ corresponding to $v_1$ and $v_2$. This is possible, since these two elements of $G$ are arcwise connected and

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$^2$ The method of proof used below, which extended this lemma from manifolds $M$ with $H_1(M) = 0$ to compact spaces $X$ with $H_1(X)$ a torsion group, was suggested by the referee.

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