ON CLOSED CONVEX SURFACES

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1. Introduction. The purpose of this paper is to prove the following theorem. Let $S$ and $\overline{S}$ be two closed orientable convex surfaces of class $C'''$ imbedded in an euclidian space $E^3$ of three dimensions, and possessing no parabolic points. Let $h$ be a differentiable homeomorphism of $S$ into $\overline{S}$ such that (a) $II = \overline{II}$, $II$ and $\overline{II}$ being the second fundamental forms of $S$ and $\overline{S}$ respectively, (b) such that the Gaussian curvatures $K$ and $\overline{K}$ of $S$ and $\overline{S}$ are equal at corresponding points $X$ and $\overline{X}$, and (c) such that the orientations of $S$ and $\overline{S}$ are preserved. Then $h$ is a rigid motion.

Incidental to the proof of the theorem, we present a simple proof of Liebmann's theorem on the rigidity of the sphere. In seeking an integral formula furnishing a proof of the theorem an integral formula was found which gave a simple proof of the fact that such a surface $S$, described in the theorem and for which the ratio of the mean to the Gaussian curvature is a constant, is a sphere. Such a surface is of course a "special" Weingarten surface. Chern has proved [3] that all convex special $W$-surfaces are spheres. Hence our statement is but a special case of Chern's theorem.

Since $S$ and $\overline{S}$ are orientable, we may assume that their second fundamental forms are positive definite.

2. Exterior forms on $S$. Let $0 - I_1, I_2, I_3$ be a fixed orthogonal frame in $E^3$. Let $(x^1, x^2, x^3)$ be the coordinates of a point $X$ in $E^3$ with respect to this orthogonal frame. The vector equation of the surface $S$ has the form

$$X = X(u^1, u^2),$$

wherein the components $(x^1, x^2, x^3)$ of the position vector $X$ are of class $C'''$ in a simply connected domain $D$ of a parameter plane. Moreover the vector $X_1 \times X_2$ wherein $X_a = (\partial X/\partial u^a)$, is not a null vector for any point of $D$.

We shall use the usual summation convention: repeated indices indicating summation over the range of the indices. We shall let the roman letters have the range 1, 2, 3 and the greek letters the range 1, 2.
Let $E_3$ be the unit normal vector of $S$. The first and second fundamental forms of $S$ are given by

\begin{equation}
I = (dX)^2 = X_\rho \cdot X_\sigma du^\rho du^\sigma = g_{\rho\sigma} du^\rho du^\sigma, \\
II = - dE_3 \cdot dX = d_\rho du^\rho du^\sigma.
\end{equation}

Let $X - E_1, E_2, E_3$ be a frame, to be called a conjugate frame, such that $(E_1, E_2, E_3) > 0$, and such that if

\begin{equation}
E_1 = U^\rho X_\rho, \quad E_2 = V^\rho X_\rho,
\end{equation}

then

\begin{equation}
d_\rho U^\sigma U^\sigma = 1, \quad d_\rho U^\rho V^\sigma = 0, \quad d_\rho V^\rho V^\sigma = 1,
\end{equation}

wherein $U^\sigma, V^\sigma$ are functions of $u^1 u^2$ Class $C''$. Conditions (2.4) imply that the tangent vectors $E_1, E_2$ form an orthonormal frame with respect to the metric $II$. They are conjugate vectors in the sense of Dupin.

Let

\begin{equation}
E_i = a, \quad E_1 \cdot E_2 = b, \quad E_2 = c.
\end{equation}

Let $\omega^1, \omega^2$ be two forms on $S$ defined by

\begin{equation}
\omega^1 = U_\rho du^\rho, \quad \omega^2 = V_\rho du^\rho, \quad U_\alpha = d_\alpha U^\rho, \quad V_\alpha = d_\alpha V^\rho.
\end{equation}

From (2.3) and (2.6) we find that

\begin{equation}
X_\alpha = U_\alpha E_1 + V_\alpha E_2, \quad d\omega^\alpha = U^\alpha \omega^1 + V^\alpha \omega^2.
\end{equation}

Hence

\begin{equation}
dX = X_\rho du^\rho = (U_\rho E_1 + V_\rho E_2)(U^\rho \omega^1 + V^\rho \omega^2) = \omega^i E_i, \quad \omega^3 = 0.
\end{equation}

We also write

\begin{equation}
dE_i = \omega^j E_j, \quad \omega^3 = 0.
\end{equation}

Taking exterior differentials of (2.8) and (2.9), and using (2.8) and (2.9) we find that

\begin{equation}
d\omega^i = \omega^j \wedge \omega^i, \quad d\omega_j = \omega^k \wedge \omega^i.
\end{equation}

Equations (2.10) are of course the conditions of compatibility of (2.8) and (2.9).

Using (2.4) and (2.5) we find that the first and second fundamental forms of $S$ are

\begin{equation}
I = a(\omega^1)^2 + 2b \omega^1 \omega^2 + c(\omega^2)^2, \quad II = (\omega^1)^2 + (\omega^2)^2.
\end{equation}
It follows from (2.11) that the mean $H$ of the principal normal curvatures and the Gaussian curvature $K$ of $S$ are given by

$$2H = (a + c)K, \quad (ac - b^2)K = 1. \tag{2.12}$$

Since $\omega^3 = 0$, we find from the first of (2.10) that

$$\omega^1 \land \omega^3 + \omega^2 \land \omega^3 = 0.$$ 

Hence $\omega^3$ and $\omega^3$ must have the form

$$\omega^1 = p\omega^1 + q\omega^2, \quad \omega^2 = q\omega^1 + r\omega^2. \tag{2.13}$$

Since $E_1 \cdot E_2 = 0$, $E_2 \cdot E_3 = 0$, it follows from (2.9) that

$$\omega^i E_i \cdot E_i + \omega^i E_i = 0, \quad \omega^i E_i \cdot E_i + \omega^i E_i = 0.$$ 

Hence

$$\omega^3 = - (a\omega^1 + b\omega^2), \quad \omega^3 = - (b\omega^1 + c\omega^2). \tag{2.14}$$

From (2.2) the second fundamental form of $S$ is given by

$$\Pi = - dE_3 \cdot dX = - [(a\omega^1 + b\omega^2)\omega^1 + (b\omega^1 + c\omega^2)\omega^2] = \omega^1\omega^1 + \omega^2\omega^2.$$ 

But from (2.13) and the second of (2.11) we find that $p = r = 1, q = 0$. Therefore (2.13) and (2.14) assume the form

$$\omega^1 = \omega^1, \quad \omega^2 = \omega^2, \quad a\omega^1 + b\omega^2 = - \omega^1, \quad b\omega^1 + c\omega^2 = - \omega^2. \tag{2.15}$$

From the last two of (2.15) we find that

$$\omega^3 = K(-\omega^1 + b\omega^2), \quad \omega^3 = K(b\omega^1 - a\omega^2). \tag{2.16}$$

Taking exterior differentials of the first two of (2.15) we find that

$$2\omega^1 \land \omega^1 = \omega^2 \land (\omega^2 + \omega^1), \quad 2\omega^2 \land \omega^2 = \omega^1 \land (\omega^1 + \omega^1). \tag{2.17}$$

It follows from (2.17) that $\omega^1, \omega^2, \omega^1 + \omega^2$ have the following form

$$\omega^1 = A\omega^1 + B\omega^2, \quad \omega^2 = A'\omega^1 + B'\omega^2, \quad \omega^1 + \omega^2 = 2(B\omega^1 + A'\omega^2). \tag{2.18}$$

From (2.5) and (2.9) we find that
\[ da = 2(a \omega_1^1 + b \omega_1^2), \quad dc = 2(b \omega_2^1 + c \omega_2^2), \]
\[ db = b(\omega_1^1 + \omega_2^1) + a \omega_2^1 + c \omega_1^1. \]

Hence
\[ d(ac - b^2) = 2(\omega_1^1 + \omega_2^1)(ac - b^2). \]

Therefore
\[ dK = -2(\omega_1^1 + \omega_2^1)K. \]

3. **An associated Riemannian space** \( \mathcal{R} \). Consider a Riemannian space \( \mathcal{R} \) defined over \( D \) whose metric is given by
\[ ds^2 = \Pi = (\omega_1^1)^2 + (\omega_2^1)^2. \]

We shall call this space the **associated Riemannian space**.

As is well known for Riemannian spaces [2] there exists an unique form \( \psi^1 = -\psi_1^2 \) such that
\[ d\omega^1 = \omega^2 \wedge \psi_2^1, \quad d\omega^2 = \omega^1 \wedge \psi_1^2. \]

That this form is unique follows from assuming there are two forms \( \psi_1^1, \psi_2^2 \) satisfying (3.2). That is, not only does (3.2) hold but also
\[ d\omega^1 = \omega^2 \wedge \psi_2^1, \quad d\omega^2 = \omega^1 \wedge \psi_1^2. \]

Subtracting (3.2) from the above it follows that
\[ \omega^2 \wedge (\psi_2^1 - \psi_2^2) = 0, \]
\[ \omega^1 \wedge (\psi_1^2 - \psi_1^1) = 0. \]

But since \( \omega^1, \omega^2 \) are linearly independent, \( \psi_1^2 = \psi_2^1 \). From the first of (2.10) with first \( i=1 \), then \( i=2 \) and using (2.17) we find readily that
\[ \psi_1^2 = \frac{1}{2}(\omega_1^2 - \omega_2^1). \]

satisfies (3.2) and hence is the desired unique form.

Using (2.10) we find that
\[ d\psi_1^2 = \frac{1}{2} \left[ (\omega_1^1 - \omega_2^2) \wedge (\omega_2^1 + \omega_1^2) + \omega^1 \wedge \omega_3^2 - \omega^2 \wedge \omega_3^1 \right]. \]

Hence from (2.12), (2.16) and (2.18) we find that
(3.4) \[ d\psi_1^2 = - (G + H)\omega^1 \wedge \omega^2, \]

wherein \( H \) is the mean curvature of \( S \) and

(3.5) \[ G = A'(A' - A) + B(B - B'). \]

It follows that

(3.6) \[ \mathcal{K} = G + H \]

is the Gaussian curvature of the associated Riemannian space. We shall call this curvature the associated curvature of \( S \). The associated curvatures of all surfaces having the same second fundamental forms are of course the same.

4. Integral formulas on \( S \). Before developing the formulas we shall use to prove the theorem, it will be necessary to consider the effect on the functions \( a, b, c \) and the forms \( \omega_i, \omega_j \) by a change of conjugate frame.

Let \( F' = X - E'_1, E'_2, E'_3 = E_3 \) be a second conjugate frame. Letting

\[ E'_1 = u' x_\rho, \quad E'_2 = v' x_\rho, \]

and noting that \( E'_1, E'_2 \) have the same orientation as \( E_1, E_2 \), and are also orthonormal with respect to \( \mathcal{II} \) (cf. 2.4) we may write

(4.1) \[ E'_1 = E_1 \cos \theta + E_2 \sin \theta, \quad E'_2 = -E_1 \sin \theta + E_2 \cos \theta, \]

\( \theta \) being a function of \( u^1, u^2 \) of class \( C' \).

The functions \( a', b', c' \) corresponding to \( a, b, c \) are readily found to be given by

(4.2) \[ a' = a \cos^2 \theta + b \sin 2\theta + c \sin^2 \theta, \quad 2b' = (c - a) \sin 2\theta + 2b \cos 2\theta, \quad c' = a \sin^2 \theta - b \sin 2\theta + c \cos^2 \theta. \]

Since

\[ U'^\alpha = U^\alpha \cos \theta + V^\alpha \sin \theta, \quad V'^\alpha = -U^\alpha \sin \theta + V^\alpha \cos \theta \]

it follows that, if \( \phi_i, \phi_j \) are the forms corresponding to \( \omega_i, \omega_j \),

(4.3) \[ \phi^1 = \omega^1 \cos \theta + \omega^2 \sin \theta, \quad \phi^2 = -\omega^1 \sin \theta + \omega^2 \cos \theta. \]

Moreover

(4.4) \[ \phi^1 \wedge \phi^2 + \omega^1 \wedge \omega^2. \]

Direct computation from (4.1) gives
\[\begin{align*}
\phi_1 &= \omega_1 \cos^2 \theta + \frac{1}{2} (\omega_1^2 + \omega_2) \sin 2\theta + \omega_2 \sin^2 \theta, \\
\phi_2 &= \omega_1 \sin^2 \theta - \frac{1}{2} (\omega_1^2 + \omega_2) \sin 2\theta + \omega_2 \cos^2 \theta,
\end{align*}\]

\[\begin{align*}
\phi_1 &= \frac{1}{2} (\omega_2 - \omega_1) \sin 2\theta + \omega_1 \cos^2 \theta - \omega_2 \sin^2 \theta + d\theta, \\
\phi_2 &= \frac{1}{2} (\omega_2 - \omega_1) \sin 2\theta - \omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta - d\theta, \\
\phi_3 &= \omega_3 \cos \theta + \omega_3 \sin \theta, \quad \phi_3 = -\omega_3 \sin \theta + \omega_3 \cos \theta.
\end{align*}\]

It follows from the third and fourth of (4.5) that \(d\psi_3^2\) is independent of the conjugate frame \(F\), a fact which is geometrically evident.

We find it convenient at this point to note, using (2.4) and (2.6), that

\[\omega^1 \wedge \omega^2 = U_1 V_2 d\omega^1 \wedge d\omega^2 = (U_1 V_2 - U_2 V_1) d\omega^1 \wedge d\omega^2.\]

We note (4) that

\[U_1 V_2 - U_2 V_1 = d^{1/2},\]

wherein \(d = \det (d_{\alpha \beta})\). Hence

\[\omega^1 \wedge \omega^2 = d^{1/2} d\omega^1 \wedge d\omega^2 = K^{1/2} g^{1/2} d\omega^1 \wedge d\omega^2 = K^{1/2} dA,\]

\(dA\) being “the element of area” of \(S\).

Let us now define auxiliary functions \(y_i\) by the formulas

\[y_i = X \cdot E_i.\]

We find readily that

\[dy_1 = \omega_1^1 + b\omega_1^2 + \omega_1^2 y_i, \quad dy_2 = b\omega_1^2 + c\omega_2^1 + \omega_2^1 y_i.\]

We note from (4.2) and (4.5) that the following forms are independent of the frame \(F\):

\[\begin{align*}
\omega_1 &= K^{1/2} (y_1 \omega_1^2 - y_2 \omega_1^1), \\
\omega_2 &= K^{1/2} [y_1 (b\omega_1^1 + c\omega_2^2) - y_2 (a\omega_1^2 + b\omega_2^2)], \\
\omega_3 &= K^{1/2} (\omega_1^1 - \omega_2^1), \\
\omega_4 &= K^{1/2} [2b(\omega_1^1 - \omega_2^1) + (c - a)(\omega_2^2 + \omega_2^1)].
\end{align*}\]
wherein
\[ z_1 = (a - c)y_1 + 2by_2, \quad z_2 = 2by_1 + (c - a)y_2. \]

Stokes' formula applied to a linear form \( \omega \) may be written as
\[
\int_C \omega = \iint_R d\omega,
\]
wherein \( C \) is the boundary of the simply connected region \( R \). Applying this formula to the forms (4.8) in turn, and recalling that \( S \) was assumed closed and convex, we find that
\[
\begin{align*}
\iint_S (H + Ky_3) dA &= 0, \\
\iint_S (1 + H y_3) dA &= 0, \\
\iint_S K(p + 2f^\circ y_\circ) dA &= 0, \\
\iint_S K[p + 4(a + c)G] dA &= 0,
\end{align*}
\]
(4.9)
wherein
\[
\begin{align*}
p &= a^2 + c^2 - 2ac + 4b^2, \\
f^1 &= aA + 2bB + cA', \\
f^2 &= aB + 2bA' + cB'.
\end{align*}
\]
The first and second of (4.9) are of course the familiar formulas associated with closed surfaces, the first being Minkowski's formula [2].

The first of (4.10) using (2.12) may be written in the form
\[
(4.11) \quad p = (a - c)^2 + 4b^2 = (a + c)^2 - 4(ac - b^2) = 4(H^2 - K)/K^2.
\]
Hence \( p \geq 0 \) on \( D \), the equality holding only at an umbilical point of \( S \).

Using (3.6) we may write the fourth of (4.9) in the form
\[
(4.12) \quad \iint_S H(2\mathcal{K} - H) dA = \iint_S K dA.
\]
Formula (4.12) relates the mean and associated curvatures of \( S \) with the curvatura integra, and hence to the genus of \( S \).

Consider now the first of (2.12) written in the form
\[
2H/K = a + c.
\]
From (2.18) and (2.19) we find readily that
\[
d(a + c) = 2(f^1\omega^1 + f^2\omega^2).
\]
Therefore in case the ratio of the mean to the Gaussian curvature is a constant, the third of (4.9) assumes the form
\[ \int \int_{S} K p dA = 0. \]

Hence since \( p \geq 0 \), it follows that every point of \( S \) is an umbilic, and \( S \) is a sphere.

5. **The proof of the theorem.** The formulas developed in the previous sections have analogous forms for the surface \( \overline{S} \) of the theorem. We shall denote the corresponding expressions for \( \overline{S} \) by the same but barred letters.

The homeomorphism \( h: S \rightarrow \overline{S} \) induces a homeomorphism \( h^* \) on contravariant tensors at a point \( X \) of \( S \) and a homeomorphism \( h_* \) on covariant tensors at a point \( X \) of \( S \) into contravariant and covariant tensors respectively at \( \overline{X} = hX \) on \( \overline{S} \).

Let \( F = X - E_1, E_2, E_3 \) be a given conjugate frame of \( S \). We take for the frame of \( \overline{S} \), the image of \( F \), that is
\[
\bar{F} = hX - h^*E_1, h^*E_2, E_3,
\]
\( E_3 \) being the unit normal vector of \( S \). This frame \( \bar{F} \) is a conjugate frame on \( \overline{S} \) since the form \( \Pi \) is preserved under \( h \). Using such corresponding frames on \( S \) and \( \overline{S} \), it follows that
\[
(5.1) \quad \bar{\omega}^i = h_*\omega^i.
\]

By assumption \( K = \overline{K} \), hence from (2.20)
\[
\bar{\omega}^1 + \bar{\omega}^2 = \omega^1 + \omega^2.
\]

From (2.18) and its analogue for \( \overline{S} \), it follows that
\[
(5.2) \quad \bar{A} + \bar{A}' = A + A', \quad \bar{B} + \bar{B}' = B + B'.
\]

Since \( \Pi = \Pi' \), from (3.1), (3.3) and (3.5) it follows that
\[
(5.3) \quad \bar{\omega}^2 - \bar{\omega}^1 = \omega^2 - \omega^1.
\]

Moreover from (4.2) and the analogue of (4.5) for \( \overline{S} \), the form
\[
\bar{\omega} = K^{1/2} \left[ 2b(\omega^1 - \omega^2) + (c - a)(\bar{\omega}^1 + \bar{\omega}^2) \right]
\]
is independent of the frame \( F \), and hence is a meaningful linear form on \( S \). Using (5.1), (5.2) and (5.3) we find that
\[
(5.4) \quad d\bar{\omega} = K[C'K + 4(a + c)G']dA,
\]
wherein
\[ C' = 2\delta + 4\left(H\overline{H} - K\right)/K^2, \]
\[ 2G' = 2(A'\overline{A}' + B\overline{B}) - \overline{AA}' - A\overline{A}' - B\overline{B}' - \overline{BB}', \]
\[ \delta = \begin{vmatrix} a - \overline{a} & b - \overline{b} \\ b - \overline{b} & c - \overline{c} \end{vmatrix}. \]

Application of Stokes' formula to the form \( \phi \) over the closed surface \( S \) gives
\[ \int \int_S K\left[C'K + 4(a + c)G'\right]dA = 0. \] (5.5)

Subtracting the last of (4.9) from (5.5), we obtain on using (2.12) and (3.6)
\[ \int \int_S \left[K^2\delta + 2H(2G' - G - \overline{G})\right]dA = 0. \] (5.6)

Defining \( \Delta \) by the formula
\[ \Delta = \begin{vmatrix} A - A' - (\overline{A} - \overline{A}') & B - B' - (\overline{B} - \overline{B}') \\ B - \overline{B} & A' - \overline{A}' \end{vmatrix}, \] (5.7)
we find that
\[ \Delta = 2G' - G - \overline{G}. \]

Hence (5.6) assumes the form
\[ \int \int_S (K^2\delta + 2H\Delta)dA = 0. \] (5.8)

Since \( \overline{\delta} - \overline{\delta} = ac - b^2 \), it is known [1] that \( \delta \leq 0 \) over \( D \), the equality holding if and only if \( a = \overline{a}, b = \overline{b}, c = \overline{c} \). Moreover use of (5.2) enables us to write (5.7) in the form
\[ \Delta = -2 \begin{vmatrix} A' - \overline{A}' & \overline{B} - B \\ B - \overline{B} & A' - \overline{A}' \end{vmatrix}. \]

It follows that \( \Delta \leq 0 \). From the fact that \( \delta \leq 0 \) over \( D \), and from (5.8) it follows that
\[ \int \int_S H\Delta dA \geq 0, \]
and from \( \Delta \leq 0 \), that
\[ \int \int_S H\Delta dA \leq 0. \]
Hence
\[ \iint_S H \Delta dA = 0; \text{ therefore } \iint_S K^2 \delta dA = 0. \]
Hence \( \delta = 0 \), and the first fundamental forms of \( S \) and \( \overline{S} \) are the same. Hence the homeomorphism \( h \) is an isometry and therefore a rigid motion, as was to be proved.

Suppose that the Gaussian curvature \( K \) is a constant. From (2.20) and (2.18) it follows that
\[ A + A' = 0, \quad B + B' = 0. \]
We may write (3.5) in the form
\[ G = 2(A^2 + B^2) \geq 0. \]
Since \( p \geq 0 \), \( G \geq 0 \) it follows from the last of (4.9) that
\[ G = 0, \quad p = 0. \]
Hence \( S \) is a sphere. This furnishes the promised simple proof of Liebmann's theorem.

We observe that if the associated curvature of \( S \) is equal to its mean curvature then \( G = 0 \), and \( S \) is a sphere.

References

2. ———, *Introduction to differential geometry* (mimeographed notes), the University of Chicago.

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