NOTES ON LINEARLY COMPACT ALGEBRAS

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Let $A$ be a linearly compact ring with ideal neighborhoods of zero, and let $N$ be its radical. Zelinsky shows that [3, Theorem 1] $A/N$ is algebraically and topologically isomorphic to a complete direct sum (i.e., a cartesian product) of discrete simple rings with minimum condition. In case $A$ is commutative, then [3, Theorem 2] $A$ is algebraically and topologically isomorphic to a complete direct sum of a radical ring and primary rings with units, all the summands being linearly compact. If $A$ is an algebra and the closure of powers of $N$ has zero intersection, he then shows [4, Theorem C, p. 320] that $A$ (having the usual properties) satisfies the Wedderburn principal theorem. The restriction of $N$ is needed at two stages: raising of orthogonal idempotents of $A/N$ to orthogonal idempotents of $A$, and the inductive process of producing the semi-simple part. We propose to show that, if $A$ is commutative, the Wedderburn principal theorem is valid without restriction on $N$. The problem of raising orthogonal idempotents no longer exists, for idempotents which are orthogonal modulo $N$ are already orthogonal; indeed to each idempotent in $A/N$ there is only one idempotent representative in $A$. By [3, Theorem 2] we can restrict ourselves to primary algebras. Then $A/N$ is a field and we may avail ourselves of the results of field theory to construct the semi-simple part. Our main tool (Lemma 1) is a result of Jacobson [2, Theorem 6]. It also follows easily from this that we can raise a countable number of idempotents with no restriction on the radical. We use the terminology of [3].

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1. We need the following result which is a slight modification of [2, Theorem 6] to topological rings with ideal neighborhoods of zero. Its proof is effectively the same as that of [2, Theorem 6].

**Lemma 1.** Let $A$ be a topological ring with ideal neighborhoods of zero. Let $I$ be a closed subring contained in the radical of $A$. Then, for each $a \in I$, $a = 0$ if the closure of $aI$ is $I$.

Let $A$ be a linearly compact commutative primary algebra with

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unit over a field $K$. Then $A/N$ is a field extension of $K$. Every element $u$ not in the radical has an inverse, for it has an inverse $v$ modulo $N$ and $uv$ has an inverse by the definition of the radical. We assume that $A/N$ is separable over $K$, i.e., there is a transcendence base $(\alpha_i)_{i \in I}$ such that $A/N$ is separably algebraic over $K((\alpha_i)_{i \in I})$.

**Theorem 1.** Let $A$ be a linearly compact commutative primary algebra with unit over a field $K$, and let $N$ be its radical. Suppose $A/N$ is separable over $K$. Then $A = S \oplus N$, where $S$ is a closed subalgebra isomorphic to $A/N$.

**Proof.** Let $(\alpha_i)_{i \in I}$ be a transcendence base of $A/N$ over $K$, and let $(\alpha_i)_{i \in I}$ be a set of representatives in $A$. Then no polynomials in $(\alpha_i)_{i \in I}$ with coefficients in $K$ will take values in $N$. Hence, $A$ contains the field $F$ generated by $(\alpha_i)_{i \in I}$ over $K$, and $A$ may be considered as an algebra over $F$. Therefore, we may assume that $A/N$ is separably algebraic over $K$.

First consider the case that $A/N$ is finite over $K$. Then $A/N = K(\theta)$ for some element $\theta \in A/N$. Let $a$ be a representative of $\theta$. Denote by $K[a]$ the polynomial ring in $a$. $K[a]$ will be the desired algebra if $f(a) = 0$, where $f$ is the minimal polynomial of $\theta$. We find such an $a$ as follows. Consider the collection of all subvarieties $a + I$, where $a$ is a representative of $\theta$ and $I$ is a closed ideal containing $f(a)$. Partially order this collection by set inclusion: $a + I > a' + I'$ if and only if $a + I \subset a' + I'$. Because of linear compactness, every linearly ordered subset of this collection has a least upper bound. Hence there is a maximal element, $a + I$ say. We claim $f(a) = 0$. Suppose $f(a) = n \neq 0$. Since $A/N$ is separably algebraic over $K$, $f'(a) \not\equiv 0 \pmod{N}$, where $f'$ denotes the formal derivative of $f$. Let $b$ be the inverse of $f'(a)$ and $a' = a - bn$. Then $f(a') = f(a - bn) = f(a) - f'(a)bn \equiv 0 \pmod{nI}$. By Lemma 1, $a' + I' \supset a + I$, where $I'$ is the closure of $nI$. This contradicts the maximality of $a + I$.

In general, let $S$ be a maximal subfield contained in $A$. If $A/N \neq S$ then we may extend $S$ to a field $S(a)$ by above paragraph. Hence $A = S \oplus N$. Since $N$ is the unique maximal ideal of $A$, $S$ is a closed subalgebra of $A$.

**Remarks.** (1) The subfield $S$ is unique if it is algebraic over $K$. Suppose $A = S \oplus N = S' \oplus N$. Take an $s \in S$, $s = s' + n$ with $s' \in S'$ and $n \in N$. Let $f$ be the minimal polynomial of $s + N = s' + N$. Then $0 = f(s) = f(s' + n) = f(s') + f'(s')n + \cdots$. Since $f(s') = 0$ and $f'(s') \neq 0$, $n = 0$ by Lemma 1. It is clear that if $A/N$ is not algebraic over $K$ then $S$ is no longer unique.

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(2) If $A$ is a primary ring with unit and of characteristic zero (i.e., $kx=0$ implies $x=0$, where $k$ is any positive integer), then Theorem 1 is still valid. For in this case $A$ can be considered as an algebra over the field of rational numbers.

(3) If $A/N$ is finite over $K$, the commutativity of $A$ enters only in $A/N$; when we deal with $A$ we only consider elements of the form $a, f(a), f'(a), \ldots$. If only the commutativity of $A/N$ is assumed, we do not know whether $S$ is still unique.

Theorem 1 together with [3, Theorem 2] yields:

**Theorem 2.** Let $A$ be a linearly compact commutative algebra over a field $K$, with ideal neighborhoods of zero. Then [3, Theorem 2] $A$ is algebraically and topologically a complete direct sum of primary algebras with unit and a radical algebra, each summand being linearly compact. Suppose that the quotient algebra of each primary summand over its radical is separable over $K$. Then $A$ contains a closed subalgebra $S$ such that $A=S \oplus N$ (vector space direct sum), where $N$ is the radical of $A$. Moreover, $S$ is unique if each quotient algebra is algebraic over $K$.

2. The condition that the intersection of powers of $N$ be zero enters into the task of raising idempotents only when we want to raise any number of them. If we are willing to restrict ourselves to a countable number of idempotents then linear compactness alone will do, as is shown in the following lemma. From this we get an analogue of [1, Theorem 1].

**Lemma 2.** Let $A$ be a linear compact ring with ideal neighborhoods of zero, and $N$ its radical. Then a countable number of orthogonal idempotents can be raised to orthogonal idempotents in $A$.

**Proof.** It suffices to consider two idempotents $\bar{e}, \bar{f}$ in $A/N$, $\bar{e}\bar{f}=0$. Let $e$ be an idempotent in $A$ representing $\bar{e}$. If $a$ is a representative of $\bar{f}$, then $b=a-ae+eae$ is also a representative, and $eb=be=0$. Let $I$ be the closed principal right ideal generated by $n=b^2-b$. Since $eb=0$, $eI=0$. Consider the collection of subvarieties $a+I$, where $a+N=\bar{f}, ea=ae=0$ and $I$ is the closed principal right ideal generated by $a^2-a$. Partially order this collection by set inclusion. By linear compactness there is a maximal element, $f+I$ say. Suppose $f^2-f \neq 0$. It follows from $(1-2f)^2=1+4(f^2-f) \equiv 1 \pmod{N}$, that $(1-2f)$ has an inverse.\footnote{The formal use of 1 is permissible, since it appears only in products with elements of $A$.} Let $n=(f^2-f)(1-2f)^{-1}$, $n \in I$, and let $I'$ be the closure of $nI$. Then $f'+I'>f+I$, where $f'=f+n$. This contradicts the maximality of $f+I$.\footnote{The formal use of 1 is permissible, since it appears only in products with elements of $A$.}
Theorem 3. Let $A$ be a linearly compact algebra over a field $K$ with ideal neighborhoods of zero, and let $N$ be its radical. Then [3, Theorem 1] $A/N$ is algebraically and topologically isomorphic to a complete direct sum of discrete simple rings with minimal condition. Suppose that the summands are countable in number and that each summand is a total matrix algebra over $K$. Then there is a closed subalgebra $S$ such that $A = S \oplus N$.

Proof. We need only consider one summand. Then we follow the proof of [4, Theorem C, p. 320] summing up the semi-simple parts to get the subalgebra $S$. Let, therefore, $A/N$ be a total matrix algebra over $K$ and $\delta_{ij}$ ($i, j = 1, 2, \cdots, n$) be a set of matrix units of $A/N$. The theorem will be established if we can raise $\delta_{ij}$ to a set of matrix units $e_{ij}$ of $A$. By Lemma 2 we can find all the diagonal elements $e_{ii}$. It remains to find $e_{ij}$ ($i \neq j$). It suffices to find all the $e_{ij}$ matching a given set of representatives $e_{ij}$ ($i = 2, \cdots , n$), where $e_{1i} = e_{11}e_{1i} = e_{1i}e_{1i}$. Then put $e_{ij} = e_{ii}e_{ij}$.

We construct, for instance, $e_{21}$ as follows. If $a$ is any representative of $\delta_{21}$ such that $a = e_{22}a = ae_{11}$, and if $ae_{12} = e_{22}$ then $e_{12}a = e_{11}$, for $e_{11} - e_{12}a$ is an idempotent in $N$. Consider the collection $a + I$, where $a$ is a representative of $\delta_{21}$ with $a = e_{22}a = ae_{11}$ and $I$ is a closed right ideal containing $e_{22} - ae_{12}$. Partially order this collection by set inclusion. By linear compactness there is a maximal element $e_{21} + I$. We wish to show $e_{21}e_{12} = e_{22}$. Suppose $e_{22} - e_{21}e_{12} = n \neq 0$. Then $e_{21} + (nI)^{-1}e_{21} + I$, where $e_{21}' = 2e_{21} - e_{21}e_{12}e_{21}$, contradicting the maximality of $e_{21} + I$.

Let $S$ be the total matrix algebra generated by $e_{ij}$. Since $N$ is the unique maximal ideal, $S$ is closed. We have $A = S \oplus N$.

Bibliography


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