ON QUASI-ORTHOGONAL POLYNOMIALS

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In the course of a proof, A. Rosenthal [2] used a class of polynomials which, upon specialization of certain constants, may be defined by

\begin{equation}
R_n^r(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} x^n e^{-\frac{x^2}{2}} \quad (n, \nu = 0, 1, 2, \cdots).
\end{equation}

These polynomials can be expressed in terms of the Hermite polynomials as follows:

\begin{equation}
R_n^r(x) = 2^{-r} \sqrt{\pi} \sum_{k=0}^{[\nu/2]} \frac{H_{n+\nu-2k}(x)}{(\nu - 2k)!k!}.
\end{equation}

From this relation and the well-known orthogonality of the Hermite polynomials, it follows that these polynomials satisfy the "quasi-orthogonality" relations

\begin{equation}
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} R_m^r(x) R_n^r(x) dx = 0 \quad \text{for} \quad m \neq n \pm 2j \quad (j = 0, 1, \cdots, [\nu/2]).
\end{equation}

The preceding admits of an obvious generalization. We will consider real polynomials although, with appropriate notational changes, complex variables could be considered.

Let \( p_n(x) \) denote the \( n \)th orthonormal polynomial associated with a distribution \( d\alpha(x) \) on an interval \((a, b)\). Let \( k \) and \( r \) be fixed integers, \( k \geq 0 \), \( r \geq 1 \). Define \( p_{-m}(x) = 0 \) for \( m = 1, 2, \cdots, kr \). Then the polynomials

\begin{equation}
q_n(x) = \sum_{j=0}^{k} a_{n,j} p_{n-jr}(x) \quad (a_{n,j} \text{ constants, } a_{n,0} \neq 0)
\end{equation}

clearly satisfy the relations

\begin{equation}
(q_m, q_n) = \int_{a}^{b} q_m(x) q_n(x) d\alpha(x) = 0 \quad \text{for} \quad m \neq n \pm jr
\end{equation}

\text{(j = 0, 1, \cdots, k).}

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1 This paper is an extract from the author's thesis (Purdue, 1955) which was written under the direction of Professor Arthur Rosenthal.
We will call polynomials satisfying (3), *quasi-orthogonal* polynomials of index \((k, r)\).

It is easily seen that the above is an extension of the concept of quasi-orthogonal polynomials (of order \(n+1\)) which was introduced by M. Riesz [1] and which played a fundamental role in his solution of the Hamburger moment problem. The polynomials considered by Riesz were of the form \(A_p n(x) + Bp_{n+1}(x)\), that is, of index \((1, 1)\) in the present notation.

**Theorem.** Let \(\alpha(x)\) be a nondecreasing function with infinitely many points of increase in \((a, b)\) such that the moments \(\int_{a}^{b} x^n d\alpha\) exist \((n = 0, 1, \cdots)\). Then a set of polynomials \(\{q_n(x)\}\), \(q_n(x)\) of degree \(n\), satisfy (3) if and only if they have the structure given by (2).

**Proof.** We have noted that (2) implies (3). For the converse, we take \(k > 0\) (since for \(k = 0\), we have orthogonal polynomials). Then since the orthonormal polynomials, \(p_n(x)\), associated with \(\alpha(x)\) on \((a, b)\) are uniquely determined (up to a factor of \(\pm 1\)) [3, §2.2], we can write

\[
q_n(x) = \sum_{j=0}^{n} b_{n,j}p_j(x) \quad (b_{n,n} \neq 0, n = 0, 1, \cdots).
\]

According to (3), for \(n > kr\), \(q_n(x)\) is orthogonal to every polynomial of degree \(m < n - kr\). Hence, in particular, we have \((q_n, p_m) = b_{n,m} = 0\) for \(m = 0, 1, \cdots, n - kr - 1\), and we can write

\[
q_n(x) = \sum_{j=0}^{kr} b_{n,n-j}p_{n-j}(x) \quad (n = 0, 1, 2, \cdots)
\]

where, for convenience, we define \(b_{n,n-j} = 0\) if \(n-j < 0\).

It thus remains to show that, for \(r > 1\), \(b_{n,n-j} = 0\) if \(j \neq 0, r, \cdots, kr\). To this end, let \(n > 0\) be fixed and assume this is true for every \(m < n\); that is, assume

\[
q_m(x) = \sum_{i=0}^{k} b_{m,m-ir}p_{m-ir}(x) \quad (m = 0, 1, \cdots, n - 1).
\]

Now (4) certainly holds for \(m = 0\). Suppose it failed for \(m = n\). Then there exist certain \(b_{n,n-ir+j} \neq 0\), \(0 < j < r\), \(n - ir + j \geq 0\). Let \(n - sr + t\) be the least integer such that \(b_{n,n-sr+t} \neq 0\). Then necessarily \(n - sr + t \geq 0\) and (4) implies, by virtue of (3):

\[
(q_n, q_{n-sr+t}) = b_{n,n-sr+t}b_{n-sr+t,n-sr+t} = 0.
\]

Hence we have a contradiction which establishes (4) for \(m = n\).

We next note that quasi-orthogonal polynomials satisfy a three
term recurrence formula with polynomial coefficients. For in the preceding notation, we have for $n > kr$

$$q_{n+i}(x) = \sum_{j=0}^{k} a_{n+i,j} p_{n-j+i}(x) \quad (i = -1, 0, 1).$$

If we also write the classical recurrence formula for orthogonal polynomials

$$0 = p_{n+1-s}(x) + (\alpha_{n-s} x + \beta_{n-s}) p_{n-s}(x) + \gamma_{n-s} p_{n-s-1}(x)$$

for $s = 0, 1, \ldots, kr$, we have a system of $kr+4$ equations from which the $kr+3$ polynomials, $p_m(x)$ ($m = n - kr + 1, n - kr, \ldots, n + 1$), can be eliminated. The result of this elimination is a relation of the form

$$(5) \quad A_n(x) q_{n+1}(x) + B_n(x) q_n(x) + C_n(x) q_{n-1}(x) = 0,$$

where $A_n(x)$, $B_n(x)$ and $C_n(x)$ are polynomials in $x$. By explicitly writing the eliminant of the system, it can be shown that their degrees are at most $kr$, $kr+1$ and $kr$, respectively. Further, these are the exact degrees if and only if $a_{n,k} \cdot a_{n-1,k} \neq 0$.

A similar relation holds for $n \leq kr$, the degrees of $A_n(x)$, $B_n(x)$ and $C_n(x)$ being at most $n$, $n+1$ and $n$ in this case.

For the polynomials given by (1), in the particular cases $q_{n+i}(x) = \mathcal{R}_n^\nu(x)$ for $\nu = 2$ and 3, (5) takes the forms

$$(6x^2 + n^2 - n) \mathcal{R}_{n+1}^3(x) - 2x(2x^2 + n^2 - n - 2) \mathcal{R}_n^3(x)$$

$$+ 2n(2x^2 + n^2 + n) \mathcal{R}_{n-1}^3(x) = 0,$$

$$(6x^2 + n^2 - n) \mathcal{R}_{n+2}^3(x) - 2x(6x^2 + n^2 - n - 6) \mathcal{R}_{n+1}^3(x)$$

$$+ 2(n+2)(6x^2 + n^2 + n) \mathcal{R}_n^3(x) = 0.$$