COMPACTNESS OF THE STRUCTURE SPACE OF A RING

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Jacobson [1] has shown that a topology may be defined on the set $S_A$ of primitive ideals of any nonradical ring $A$. With this topology $S_A$ is called the structure space of $A$. The topology is given by defining closure: if $T = \{p\}$ is a set of primitive ideals then $\bar{T}$ is the set of primitive ideals which contain $\bigcap \{p \mid p \in T\}$. One of Jacobson’s results is that if $A$ has a unit then $S_A$ is compact. By working with open sets rather than closure we obtain a simpler proof of this fact and also a new sufficient condition for compactness: every 2-sided ideal of $A$ is finitely generated.

Let $p, q, \ldots$, be points of $S_A$ (primitive ideals). For each $x \in A$ write $a(x)$ for the principal (2 sided) ideal generated by $x$, and let $U_x = \{p \mid p \nsubseteq a(x)\}$.

**Lemma 1.** $\{U_x\}_{x \in A}$ is a basis of the topology.

**Proof.** Since $U_x$ is the complement of $\{p \mid p \nsubseteq a(x)\}$ and the latter set is clearly closed, all the $U_x$ are open. Let $U$ be an open subset of $S_A$, let $F = cU$, and take $p \in U$. Since $F = F = \{q \mid q \nsubseteq \bigcap F\}$ and $p \in F = F$, we have $p, F \cap F$, where $F = \{q \mid q \in F\}$. Hence $\exists a \in A$ such that $a$ belongs to the set-theoretic difference $F - p$, and $p \in U_a$. If $q \nsubseteq (a)$ then $q \nsubseteq F$, so that $q \in F$, or $q \in cF = U$. Hence $p \in U_a \subseteq U$.

**Theorem 1.** If $A$ has a unit then $S_A$ is compact.

**Proof.** We prove that any basic open cover has a finite subcover. Let $U = \{U_\mu = \{p \mid p \nsubseteq (a_\mu)\}\}$ be a basic open cover. Then $S_A = \bigcup_\mu U_\mu = \{p \mid \exists \nu \text{ such that } p \nsubseteq (a_\nu)\} = \{p \mid p \nsubseteq \sum_\mu (a_\mu)\}$. Write $I = \sum_\mu (a_\mu)$. In a ring with unit every 2 sided ideal can be imbedded in a primitive ideal [2]. But $\{p \mid p \nsubseteq I\} = S_A$ exhausts all primitive ideals. Hence $I = A$, so that the unit 1 is in $I$. Hence $\exists b_1, \ldots, b_n$ in $(a_{\mu_1}) + \cdots + (a_{\mu_n})$ such that $1 = b_1 + \cdots + b_n$, so that $A = (a_{\mu_1}) + \cdots + (a_{\mu_n}) = I$. But this means that $S_A = \{p \mid p \nsubseteq \sum_1^n (a_{\mu_i})\} = \bigcup_1^n U_{\mu_i}$.

The converse of Theorem 1 is false, as is shown by the ring $2J$ of even integers. It is easy to see that the open sets in the structure space of $2J$ are complements of finite sets (the proof in [1] of this same fact for the ring $J$ of all integers carries over mutatis mutandis). Hence if $\{G_\alpha\}$ is an open cover, then $G_1$, say, misses only finitely many points, so that $\{G_\alpha\}$ possesses a finite subcover.

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Theorem 2. If every (2 sided) ideal of $A$ is finitely generated then $S_A$ is compact.

Proof. With the notation as in Theorem 1, let $\mathcal{U} = \{U_\mu\}$ be a basic open cover. Then $S_A = \bigcup_\mu U_\mu = \{ p \mid p \not\in \sum_\mu (a_\mu) \}$. Write $I = \sum_\mu (a_\mu)$, and let $b_1, \cdots, b_n$ be a basis of $I$, so that $I = \sum_i (b_i)$. Each $b_i$ lies in an ideal generated by finitely many of the $a_\mu$, and since there are finitely many $b_i$, the set $\{b_1, \cdots, b_n\}$ is generated by finitely many $a_\mu$, say $a_{\mu_1}, \cdots, a_{\mu_r}$. That is, $I = \sum_{j=1}^r (a_{\mu_j})$. But this means that $S_A = \{ p \mid p \not\in \sum_{j=1}^r (a_{\mu_j}) \} = \bigcup_{j=1}^r U_{\mu_j}$, so that $\mathcal{U}$ has a finite subcover.

We conclude with a comment which may be regarded as a partial converse to Theorem 2. If $S_A$ is compact then as far as the structure space goes $A$ may as well have been finitely generated. The precise theorem is

Theorem 3. If $S_A$ is compact then there exists a finitely generated (2 sided) ideal $I$ of $A$ such that $S_A$ is homeomorphic to $S_I$.

Proof. Here $S_I$ means the structure space of $I$ as a ring. By a theorem of Kaplansky [3] $S_I$ is homeomorphic to $\{ p \mid p \not\in I \}$, for any ideal $I$ of $A$. We have $S_A = \bigcup_a U_a = \bigcup_{i=1}^r U_{a_i}$ by compactness. Write $I = (a_i) + \cdots + (a_n)$. Then $S_A = \{ p \mid p \not\in I \}$ and this is homeomorphic to $S_I$ by the quoted theorem.

Bibliography

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