CONVEX SETS AND NEAREST POINTS

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1. Introduction. A well known theorem due to Motzkin [11] and extended by Busemann [3] and Jessen [7] characterizes the convexity of a closed set $S$ in Euclidean $n$-space $E^n$ in the following manner: $S$ is convex if and only if to each point in $E^n$ there corresponds a unique nearest point in $S$. We show here that if $S$ is convex, the set $S_z$ of all points having $z$ as a nearest point in $S$ is a convex cone with vertex $z$, while the hypothesis that $S_z$ be merely a cone with vertex $z$ (for each $z \in S$) is shown to characterize the convexity of $S$.

In trying to establish these results in a more general normed linear space $E$ we find that the statement "$S_z$ is convex whenever $S$ is convex" is equivalent to the existence of an inner product in $E$ when the dimension of $E$ is at least three, while in a two-dimensional space it is equivalent to strict convexity. A theorem by Motzkin [12] leads easily to the analogous result that $E$ is an inner product space if and only if $S_z$ is convex for every $S \subseteq E$ (and $z \in S$).

In the concluding section we consider a nearest-point map $f$ which assigns to each point of $E$ a nearest point in a given closed set $S$. It is shown that the property "$f$ shrinks distances whenever it exists for a closed convex set" characterizes inner product spaces of three or more dimensions. In two-dimensional spaces this property is equivalent to strict convexity and symmetry of Birkhoff's orthogonality [2]. The convexity of a closed set in $E^n$ is shown to be characterized by the fact that its nearest-point map shrinks distances.

2. Definitions and remarks. Throughout this paper $E$ will be a normed linear space and $S$ a subset thereof. For $z \in S$, $S_z$ will be the set $\{x: \|x-z\| = \inf_{y \in S} \|x-y\|\}$, the set of all points in $E$ having $z$ as a nearest point in $S$. It may well be that $z$ is the only point in $S_z$. It is not difficult to verify that if a sequence of points in $S_z$ converges then its limit is in $S_z$; hence $S_z$ is always closed.

We will say that $S$ is proximinal if for each point $x$ in $E$ there is a
point of $S$ nearest to $x$, i.e., if for each point $x \in E$ there is at least one point $z$ in $S$ such that $x$ is in $S_z$. If there is a unique such $z$ for each $x$ in $E$ we will say that $S$ is **uniquely proximinal**. Although we are not primarily interested in conditions which guarantee that $S$ is proximinal we list some of the known ones:

(i) $S$ is proximinal if it is compact.
(ii) Every closed set $S$ is proximinal if $E$ is finite dimensional.
(iii) Every closed convex set $S$ is proximinal if $E$ is reflexive.

3. **Cones, smoothness and strict convexity.** The following lemma has appeared before [10] but will be proved here since we use it several times.

**Lemma 3.1.** If $S$ is convex and $z \in S$ then $S_z$ is a cone with vertex $z$.

**Proof.** We can suppose $z = \phi$. It will suffice to show that if $y \in S_\phi$, then $\lambda y \in S_\phi$ for each $\lambda > 0$. Suppose $x \in S$, then $\|y\| \leq \|y - x\|$. If $\lambda < 1$, $\|\lambda y\| + \|y - \lambda y\| = \|y\| \leq \|y - x\| \leq \|y - \lambda y\| + \|\lambda y - x\|$, whence $\|\lambda y\| \leq \|\lambda y - x\|$. If $\lambda > 1$, $\lambda^{-1}x$ is in the convex set $S$ so $\|\lambda y\| = \lambda \|y\| \leq \lambda \|\lambda y - x\| = \|\lambda y - x\|$. Hence $\lambda y \in S_\phi$ for each $\lambda > 0$.

A set $E$ is strictly convex if the boundary of its unit cell contains no line segment, i.e., if $\|x\| = 1 = \|y\|$ and $\lambda \in [0, 1]$ imply $\|\lambda x + (1 - \lambda)y\| < 1$. The following lemma characterizes strict convexity in terms useful to us.

**Lemma 3.2.** The following statements are equivalent:

(i) $E$ is strictly convex.
(ii) For each convex set $S$ and distinct points $x$ and $y$ of $S$, $S_x \cap S_y$ is empty.
(iii) Whenever a convex set $S$ is proximinal it is uniquely proximinal.

**Proof.** (i) $\Rightarrow$ (ii). If $S_x \cap S_y$ is nonempty we can suppose $\phi \in S_x \cap S_y$, so $\|x\| = \|y\|$. Now, $(1/2)(x+y) \in S$ and if $E$ is strictly convex $\|(1/2)(x+y)\| < \|x\| = \|y\|$, a contradiction.

(ii) $\Rightarrow$ (iii). This is immediate from the definitions.

(iii) $\Rightarrow$ (i). Suppose $E$ is not strictly convex, then there exist distinct points $x$ and $y$ such that $\|\lambda x + (1 - \lambda)y\| = 1$ for each $\lambda \in [0, 1]$. The (compact) convex line segment $[x, y]$ is proximinal but not uniquely proximinal since the origin is equidistant from all points of $[x, y]$.

We say that $E$ is **smooth** if its unit cell has a unique supporting hyperplane at each of its boundary points. (A hyperplane is a maximal proper closed linear variety.) We give a partial converse to Lemma 3.1 in the following lemma (stated but not proved in [10]).

**Lemma 3.3.** Suppose $S$ is closed and proximinal and $E$ is smooth. Then $S$ is convex if for each $z \in S$, $S_z$ is a cone with vertex $z$. 
Suppose $S$ is not convex. Then there exist points $u$ and $v$ in $S$ such that $[u, v] \subseteq E\sim S$. Let $x = (1/2)(u + v)$. Since $S$ is proximinal $x \in S_z$ for some $z \in S$ and we can suppose without loss of generality that $z = \phi$. Let $H$ be the unique hyperplane supporting $N_{\|x\|}x = \{y : \|y - x\| < \|x\|\}$ at $\phi$ and let $H'$ be the open halfspace determined by $H$ which contains $N_{\|x\|}x$. Not both $u$ and $v$ are in the convex set $E\sim H'$ since $x$ is not. Suppose $u \in H'$. We will show that for some $\lambda > 0$, $u \in N_{\|x\|}x = \lambda N_{\|x\|}x$ so that $\|\lambda x\| > \|\lambda x - u\|$ and hence $\lambda x \in S_\phi$, contradicting the assumption that $S_\phi$ is a cone with vertex $\phi$.

Suppose that for every $\lambda > 0$, $u \in N_{\|x\|}x$. Then $\lambda u \in N_{\|x\|}x$ for each $\alpha \in [0, 1]$ and hence the convex set $[u, \phi]$ is disjoint from $N_{\|x\|}x$. There exists a hyperplane $G$ separating $[u, \phi]$ from $N_{\|x\|}x$ which necessarily supports $N_{\|x\|}x$ at $\phi$. But $u$ is in the closed halfspace determined by $G$ which does not contain $N_{\|x\|}x$ and hence $G \neq H$, contradicting the fact that $E$ is smooth. Thus for some $\lambda > 0$, $u \in \lambda N_{\|x\|}x$, which was to be shown.

The assumption of smoothness of $E$ in Lemma 3.3 is a necessary one, since it is not difficult to show that the statement of the lemma, with smoothness omitted, implies that $E$ is smooth.

Since every closed subset of (smooth) $E^n$ is proximinal, Lemmas 3.1 and 3.3 combine to prove the following characterization of convexity.

**Theorem 3.4.** A closed set $S$ in $E^n$ is convex if and only if for each $z \in S$, $S_z$ is a cone with vertex $z$.

4. Convex cones and inner products. We say that $E$ is an inner product space if $E$ admits an inner product $(x, y)$ such that $\|x\| = (x, x)^{1/2}$. If $E$ is an inner product space it is easily verified that for points $x, y$ and $z$ in $E$ and $\lambda \in \mathbb{R}$ the following identity holds:

$$\|z - [\lambda x + (1 - \lambda)y]\|^2 = \lambda\|x - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|z - y\|^2. \tag{1}$$

(In fact [8] $E$ admits an inner product if and only if (1) holds true.)

In particular, we can see that an inner product space is strictly convex by setting $z = \phi$, $\|x\| = 1 = \|y\|$ and $\lambda \in [0, 1]$ in (1).

**Lemma 4.1.** If $E$ is an inner product space and $z \in S$ then $S_z$ is convex.

**Proof.** Suppose $z = \phi$ and that $x \in S_\phi$ and $y \in S_\phi$. Let $w = \lambda x + (1 - \lambda)y$, $\lambda \in [0, 1]$. Then if $v$ is any point of $S$, $\|x\| \leq \|x - v\|$ and $\|y\| \leq \|y - v\|$ while, by (1), we have

$$\|w\|^2 = \lambda\|x\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|y\|^2 \leq \lambda\|x - v\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|y - v\|^2 = \|w - v\|^2,$$
so $\|w\| \leq \|w-v\|$ and hence $w \in S_*$. Motzkin [12] has proved the following interesting result: Suppose that $E$ is two-dimensional. Then $E$ is an inner product space if and only if for each set $S$ and $z \in S$, $S_z$ is convex. Since a normed linear space is an inner product space if and only if each two-dimensional subspace has an inner product [8], Motzkin's result leads easily to the sufficiency portion of the following theorem.

**Theorem 4.2.** A normed linear space $E$ is an inner product space if and only if for each set $S \subseteq E$ and $z \in S$, $S_z$ is convex.

Demanding convexity of $S_z$ only when $S$ itself is convex leads to the following result, closely related to Theorem 4.2.

**Theorem 4.3.** Suppose that the dimension of $E$ is at least three [resp. equal to two]. Then $E$ is an inner product space [resp. strictly convex] if and only if for each convex set $S$ and $z \in S$, $S_z$ is convex.

The portions of Theorem 4.3 which are as yet unproved follow from the next three lemmas. The idea used in the proof of the following lemma is due to James [6, Theorem 2].

**Lemma 4.4.** Suppose the dimension of $E$ is at least three. Then $E$ is an inner product space provided $S_z$ is convex for each convex set $S \subseteq E$ and $z \in S$.

**Proof.** If $x_1$ and $x_2$ are any two linearly independent points of $E$ there exist hyperplanes $H^1$ and $H^2$ such that $x_1 \in H^1_\phi$ and $x_2 \in H^2_\phi$ ($H^i_\phi = h^{-1}_i(0)$, where $h_i$ is a continuous linear functional such that $h_i(x_i) = \|x_i\|$ and $\|h_i\| = 1$ [1, p. 55]). By Lemma 3.1, $H^1_\phi$ and $H^2_\phi$ are convex cones with vertex $\phi$ and hence $\alpha x_i \in H^1_\phi$ if $\alpha_i \geq 0$ ($i = 1, 2$). But $y \in H^1$ if and only if $-y \in H^1$, so $\|(-x_1) - y\| = \|x_1 - (-y)\| \geq \|x_1\| = \|-x_1\|$, which shows that $-x_1 \in H^1_\phi$ and therefore $\alpha x_1 \in H^1_\phi$ for $\alpha \in \mathbb{R}$. Similarly, $\alpha x_2 \in H^2_\phi$ for $\alpha \in \mathbb{R}$. Thus, if we let $G = H^1 \cap H^2$, $G_\phi$ is also a convex cone with vertex $\phi$ and hence contains $\alpha_1 x_1 + \alpha_2 x_2$ for $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$. Since $H^1$ and $H^2$ are hyperplanes, $E$ is the direct sum of $G$ and the two-dimensional sub-space $F$ spanned by $x_1$ and $x_2$. Therefore, if $z \in E$, $z = (\alpha_1 x_1 + \alpha_2 x_2) - g$, where $-g \in G$, and $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$. Letting $f(z) = \alpha_1 x_1 + \alpha_2 x_2$ we see that $f$ is a projection onto $F$ and since $f(z) \in G_\phi$ and $g \in G$, $\|f(z)\| \leq \|f(z) - g\| = \|z\|$ and therefore $\|f\| = 1$.

Thus, we can always find a projection of norm one on any two-dimensional subspace of $E$ and therefore it is possible to define an inner product in any three-dimensional subspace of $E$ [9, Theorem 3]. This, however, implies that we can define an inner product in $E$ itself [8].

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By a less direct argument than the above it can be shown that the conclusion still holds if the hypothesis "$S_z$ is convex whenever $z \in S$ and $S$ is convex" be replaced by "$L_z$ is convex whenever $L$ is a line and $z \in L$.”

**Lemma 4.5.** If $L_z$ is convex for each line $L$ and $z \in L$ then $E$ is strictly convex.

**Proof.** If $E$ is not strictly convex there exist distinct points $x$ and $y$ such that $\|\lambda x + (1 - \lambda)y\| = 1$ for each $\lambda \in [0, 1]$. Let $L = \{\lambda x + (1 - \lambda)y : \lambda \in \mathbb{R}\}$. If $\lambda > 1$, $\|\lambda x + (1 - \lambda)y\| \geq \lambda \|x\| - (1 - \lambda)\|y\| = 1$ while if $\lambda < 0$, $\|\lambda x + (1 - \lambda)y\| \geq (1 - \lambda)\|y\| - \lambda\|x\| = 1$. Thus, $(1/2)x \in L_{(1/2)(x+y)}$ since if $z$ is any point of $L$, $\|z - (1/2)x\| \geq \|z\| - (1/2)\|y\| = (1/2)\|x\| - (1/2)\|y\|$. Similarly, $(1/2)y \in L_{(1/2)(x+y)}$. Further, $x + (1/2)y \in L_{(1/2)(x+y)}$, for if $z \in L$, then $x + y - z \in L$ and hence

$$
\|((1/2)x - (1/2)(x + y))\|
= \|((1/2)x)\| = \|((1/2)x) - (1/2)y\|
\leq \|((x + y - z) - (1/2)y\| = \|((x + (1/2)y) - z\|
$$

Since $L_{(1/2)(x+y)}$ is assumed to be convex,

$$(1/2)[x + (1/2)y] + (1/2)[(1/2)x] = (3/4)x + (1/4)y \in L_{(1/2)(x+y)},$$

which is impossible, $(3/4)x + (1/4)y$ itself being a point of $L$.

**Lemma 4.6.** Suppose $E$ is strictly convex and of dimension two. Then if $S$ is convex and $z \in S$, $S_z$ is a convex cone with vertex $z$.

**Proof.** By Lemma 3.1, $S_z$ is a cone with vertex $z$, so it remains only to show that $S_z$ is convex. Suppose $z = \phi$ and suppose $x \in S_{\phi}$ and $y \in S_{\phi}$; we must show that $[x, y] \subset S_{\phi}$.

Let $K$ be the closed convex cone generated by all the rays passing from $\phi$ through points of $[x, y]$. Then $K \cap S = \{\phi\}$, for if $w \in K \cap S$ there exists $\lambda \in [0, 1]$ such that $\lambda w$ is in the closed triangle $\phi xy$ and since $S$ is convex, $\lambda w \in S$. Let $u = (\|x\| + \|y\|)^{-1}(\|y\|x + \|x\|y)$. Then $u \in [x, y]$ and hence $\lambda w$ is in the closed triangle $\phi ux$, say. (Otherwise $\lambda w$ is in $\phi uy$.) But if $\alpha u$, $\alpha \in [0, 1]$, is any point of side $[\phi, u]$, $\|x - \alpha u\| \leq \|x\|$. Consequently, $\|x - \lambda w\| \leq \|x\|$ and so $x \in S_{\lambda w}$. By strict convexity and Lemma 3.2, $\lambda w = \phi$ and therefore $w = \phi$.

Now suppose $z \in [x, y]$ and $v \in S$. Then $[z, v]$ must intersect $\{\lambda x : \lambda \geq 0\}$ or $\{\lambda y : \lambda \geq 0\}$; say $\lambda x \in [z, v]$, $\lambda \geq 0$. Then $\lambda x \in S_{\phi}$ (since $S_{\phi}$ is a cone) and $\|z\| \leq \|z - \lambda x\| + \|\lambda x\| \leq \|z - \lambda x\| + \|\lambda x - v\| = \|z - v\|$. Since this holds for arbitrary $v \in S$, $z \in S_{\phi}$.
5. The nearest-point map. If a closed set $S$ in $E$ is proximinal we can define a function $f$ from $E$ onto $S$ as follows: If $x \in E$ let $f(x)$ be a point of $S$ such that $x \in S_{f(x)}$. It is clear that $f$, called a nearest-point map for $S$, exists if and only if $S$ is proximinal, and that $f$ is unique if and only if $S$ is uniquely proximinal. We say that $f$ shrinks distances if $\|f(x) - f(y)\| \leq \|x - y\|$ whenever $x, y \in E$. We will say that $E$ has the property P if a nearest-point map shrinks distances whenever it exists for a closed convex set $S \subseteq E$. The following theorem is well known, but a proof is included for completeness.

**Theorem 5.1.** Each inner product space $E$ has the property P.

**Proof.** Suppose a nearest-point map $f$ exists for a closed convex set $S$. Since $E$ is strictly convex Lemma 3.2 implies that $f$ is unique. Suppose $x \in E$ and $y \in E$ and that $f(x) = \phi$. Let $H$ be the hyperplane through $\phi$ which is orthogonal to $f(y)$ and let $J$ be the open half-space determined by $H$ which contains $f(y)$. Let $K$ be the open half-space determined by $H + f(y)$ which contains $\phi$. If $x \in J$ there exists $\alpha > 0$ such that $\|x\| > \|\alpha f(y)\|$. Pick $\lambda > 0$ such that $\lambda \alpha = 1/2$, then $\|\lambda x\| > \|\lambda x - (1/2)f(y)\|$. But, since $(1/2)f(y) \in S$, this contradicts the fact that $f(x)$, and hence $f(\lambda x)$, is the origin. We conclude that $x \notin J$ and an entirely similar argument shows that $y \notin K$. Thus, $\|x - y\|$ is no less than the width of $J \cap K$, and this is equal to $\|f(y)\|$.

Birkhoff [2] has defined a type of orthogonality which is meaningful in a general normed linear space $E$ and which coincides with the usual notion in an inner product space. If $x \neq 0$ we say that $y$ is orthogonal to $x$ (written $y \bot x$) if $\|y - \lambda x\| \geq \|y\|$ for each $\lambda \in \mathbb{R}$. Note that this is equivalent to saying that $y \in (Rx)^\perp$, where $Rx = \{\lambda x : \lambda \in \mathbb{R}\}$ is the line determined by $x$ and $\phi$. We say that orthogonality is symmetric if $x \bot y$ implies $y \bot x$. Day [4, Theorem 6.4] and James [6, Theorem 1] have independently proved that a normed linear space of dimension at least three is an inner product space if and only if orthogonality is symmetric. We use this fact in proving the following theorem.

**Theorem 5.2.** Suppose that the dimension of $E$ is at least three [resp. equal to two]. Then $E$ is an inner product space [resp. strictly convex and orthogonality is symmetric] if and only if $E$ has the property P.

The proof is contained in Theorem 5.1 and the following succession of remarks and lemmas.

**Lemma 5.3.** If $E$ has the property P then $E$ is strictly convex.

Since the proof of this lemma is quite straightforward, it will be omitted.
Lemma 5.4. If $E$ has the property $P$ then orthogonality in $E$ is symmetric.

Proof. By Lemma 5.3, $E$ must be strictly convex and hence a nearest-point map is unique whenever it exists for a closed convex set. Suppose that neither $y$ nor $x$ is the origin and that $y \perp x$. The line $Rx$ is uniquely proximinal and the nearest-point map $f$ exists for $Rx$. Since $Ry \subseteq (Rx)_{\phi}$, $f(\lambda y) = \phi$ for any $\lambda \in R$. Now $E$ has the property $P$, so $\|x\| = \|f(x) - f(\lambda y)\| \leq \|x - \lambda y\|$ for any $\lambda \in R$, i.e., $x \in (Ry)_{\phi}$ or $x \perp y$. Thus, orthogonality is symmetric.

If the dimension of $E$ is at least three, the Day-James theorem mentioned above, together with Lemma 5.4, proves that if $E$ has property $P$ it is an inner-product space.

Lemma 5.5. Suppose that $E$ is two-dimensional. If $E$ is strictly convex and orthogonality is symmetric then $E$ has the property $P$.

Proof. Suppose the nearest-point map $f$ exists for a closed convex set $S$ and suppose $x, y \in E$. We can assume that $f(x) = \phi$. There exists a point $z \not= \phi$ such that $z \perp f(y)$ and, since $E$ is strictly convex, $w \perp f(y)$ implies $w \in Rz$ [5, Theorem 4.3]. Let $J$ be the open half-space determined by $Rz$ which contains $f(y)$ and let $K$ be the open half-space determined by $Rz + f(y)$ which contains $\phi$. If $x \in J$ there exists a unique $\alpha \in R$ such that $x - \alpha f(y) \perp f(y)$ [5]. Now, $\alpha > 0$ since $x - \alpha f(y) \in Rz$ and $x$ is on the same side of $Rz$ as is $f(y)$. Thus, using strict convexity again, $\|x - \alpha f(y)\| < \|x\|$. As in the proof of Theorem 5.1 we conclude that $x \in J$. A similar argument shows that $y \in K$. Thus, $\|x - y\|$ is no less than the width of $K \cap J$. Now, by symmetry of orthogonality, $f(y) \perp z$ and so the distance from $f(y)$ to $Rz$ is attained at $\phi$. Hence the distance from $Rz + f(y)$ to $Rz$ (which is the width of $J \cap K$) is equal to $\|f(y)\|$ and therefore $\|f(y)\| \leq \|x - y\|$. It is not hard to see that neither strict convexity nor symmetry can be omitted in Lemma 5.5, since $P$ implies both and there exist examples showing that neither implies the other.

The following theorem shows that the "shrinking" property of nearest-point maps is pretty well restricted to those which exist for convex sets.

Theorem 5.6. Suppose that $E$ is strictly convex and that a nearest-point map $f$ exists for the closed set $S \subseteq E$. Then $S$ is convex if $f$ shrinks distances.

Proof. If $S$ is not convex there exist distinct points $x$ and $y$ of $S$ such that $x, y \in E \sim S$. Letting $z = (1/2)(x + y)$ we see that one of
\[ \|x-f(z)\|, \|y-f(z)\| \] is greater than \((1/2)\|x-y\|\). (This is obvious if \(f(z)\) is \(x\) or \(y\), while if \(f(z) \neq x, y\) and neither \(\|x-f(z)\|\) nor \(\|y-f(z)\|\) is greater than \((1/2)\|x-y\|\), strict convexity implies that \(\|x-y\|<\|x-f(z)\|+\|y-f(z)\| \leq \|x-y\|\), a contradiction.) Suppose, then, that \(\|x-f(z)\| > (1/2)\|x-y\| = \|x-z\|\). Since \(f(x) = x\), this contradicts the assumption that \(f\) shrinks distances. We get the same contradiction if \(\|y-f(z)\| > (1/2)\|x-y\|\), hence \(S\) must be convex.

A simple two-dimensional example can be constructed to show that we need to assume strict convexity in the above theorem.

Since every closed subset of Euclidean \(n\)-space \(E^n\) is proximinal, Theorems 5.1 and 5.6 combine to give the following corollary.

**Corollary 5.7.** Let \(f\) be a nearest-point map for the closed set \(S \subset E^n\). Then \(S\) is convex if and only if \(f\) shrinks distances.

**References**


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