A NOTE ON THE SPAN OF TRANSLATIONS IN $L^p$

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Suppose $f \in L^1 \cap L^p$. $f$ is said to have the Wiener closure property,\(^\gamma\) (C), if the translates of $f$ span $L^p$. Since $f \in L^1$, the Fourier transform $\hat{f}$ is well defined. Let $Z(f)$ be the set of zeros of $\hat{f}$. One would like to reformulate (C) in terms of structural properties of the closed set $Z(f)$. The problem seems quite difficult; in this note we show that (C) is nearly equivalent to a uniqueness property of $Z(f)$.\(^\delta\)

It is assumed that the notion of the spectrum\(^\delta\) of a bounded continuous function is familiar.

**Definition.** A closed set is of type $U^q$ if the only bounded continuous function in $L^q$ with spectrum contained in the set is the null function.\(^\gamma\)

We shall say that $f$ has property (U) if $Z(f)$ is of type $U^q$ where $1/p + 1/q = 1$. Pollard, [4], has observed, what is true for any locally compact Abelian group, that

**Theorem 1.** For $1 \leq p < \infty$, (U) implies (C).

In the converse direction one has trivially,

**Theorem 2.** For $2 \leq p < \infty$, (C) implies (U).

Of course, this also holds for $p = 1$. What is left open is the case $1 < p < 2$. Here we have two classes of results corresponding to weakening the conclusion and strengthening the hypothesis respectively.

**Definition.** A closed set is of type $U^q^*$ if there is no nontrivial complex measure of bounded variation with spectrum (support) in the set whose Fourier-Stieltjes transform belongs to $L^q$.

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3 This is the viewpoint of [4].

4 For an elaborate treatment of spectral theory see [3]; however [5] will be more accessible to the classical analyst. For assertions about the spectrum not proved in the text see these references.

5 The definition here is equivalent to that in [4]; the proof of equivalence is essentially the same as the proofs given there. Simple modifications of the method show that sets of uniqueness can be defined using any of a large variety of summability methods, including, when $q < \infty$, ordinary convergence of trigonometric integrals.
We shall say that \( f \) has property \((U^*)\) if \( Z(f) \) is of type \( U^* \) where \( 1/p + 1/q = 1 \).

**Theorem 2*. For \( 1 \leq p < \infty \), \((C)\) implies \((U^*)\).

The only result which requires any imagination is the next. It should be noted that the proof makes only trivial use of the "natural" assumption, \( f \in L^p \), but it depends strongly on the fact that \( f \in L^1 \).

**Theorem 3.** If for some \( \epsilon > 0 \), \( f \in Lip \epsilon \), then \((C)\) implies \((U)\).

We remark that the extra hypothesis is certainly fulfilled\(^6\) if \( \int |f(x)| \, |x|^r \, dx < \infty \).

To prove the above theorems, first observe that \((C)\) is equivalent to the statement: if \( \phi \in L^q \) and the convolution \( f \ast \phi = 0 \), then \( \phi = 0 \). Let \( g \in L^1 \) be such that \( g \) vanishes outside a compact set. If \( f \ast \phi = 0 \) then \( f \ast (g \ast \phi) = 0 \) while on the other hand, \( \phi = 0 \) if and only if \( g \ast \phi = 0 \) for each such \( g \). Thus we may replace \( \phi \) if necessary by \( g \ast \phi \) and consider only bounded continuous functions \( \phi \in L^q \) with compact spectrum \( \Lambda(\phi) \). The defining property of the spectrum is that \( f \ast \phi = 0 \) implies \( \Lambda(\phi) \subseteq Z(f) \); this proves Theorem 1. The propositions in the converse direction are argued by contradiction. We assume there exists some non-null \( \phi \in L^q \) with \( \Lambda(\phi) \subseteq Z(f) \) and wish to prove that \( f \ast \phi = 0 \), or something just as good. This is essentially a spectral synthesis problem, and as such it appears to require extra conditions. For example if \( \phi \) is known to be a Fourier-Stieltjes transform, \( \Lambda(\phi) \subseteq Z(f) \) implies \( f \ast \phi = 0 \); this establishes Theorem 2*. The observation that it suffices to consider \( \phi \)'s with compact spectrum shows that for \( 1 \leq q \leq 2 \), type \( U^q \) is identical with type \( U^q* \) since every \( \phi \in L^q \) with compact spectrum is a Fourier transform. Theorem 2 is therefore an immediate corollary of Theorem 2*.

All the foregoing is valid for locally compact Abelian groups. However, for simplicity, we present the details of the proof of Theorem 3 only for the real line. The extension to the general case is clearly indicated in [3], (cf. the proof there of Lemma 4.4). Suppose \( Z(f) \) is not of type \( U^q \). Then there is a non-null \( \phi \in L^q \) with compact spectrum \( \Lambda(\phi) \subseteq Z(f) \). Let \( f^{(n)} \) denote the convolution of \( f \) with itself \( n \) times. If we can show that \( f^{(n)} \ast \phi = 0 \) for some \( n \) we are through, for let \( n \) be the first integer for which this is true. If \( n = 1 \), fine! Otherwise \( f^{(n-1)} \ast \phi \) is a non-null function in \( L^q \) with spectrum \( \Lambda(f^{(n-1)} \ast \phi) \subseteq \Lambda(\phi) \subseteq Z(f) \) and \( f \ast (f^{(n-1)} \ast \phi) = 0 \). The Lipschitz condition is just what we need to guarantee the existence of an \( n \) so that \( f^{(n)} \ast \phi = 0 \).

Choose an \( h > 0 \) and set \( k(x) = (x/2)^{-2} \sin^2 x/2 \). Define \( \Phi_h(t) \)

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\(^6\) Theorem 3 is supposed to compare favorably with Theorem B of [4].
= (2\pi)^{-1} \int \exp (-itx) k(hx) \phi(x) dx. \textrm{ Then } \Phi_h \textrm{ vanishes outside the set } \Lambda^h \textrm{ consisting of those points at a distance } <h \textrm{ from } \Lambda(\phi). \textrm{ Moreover, } f^{(n)}(x) = \int f^{(n)}(x-y) \phi(y) dy = \lim_{h \to 0} \int \exp (itx) f^{(n)}(t) \Phi_h(t) dt. \textrm{ Hence it suffices to prove that } 
abla f^{(n)}(t) \Phi_h(t) dt = o(1) \textrm{ as } h \to 0. \textrm{ Now } f \textrm{ vanishes on } \Lambda(\phi) \textrm{ and } f \in \text{Lip } \epsilon. \textrm{ Hence if } t \textrm{ is within } h \textrm{ of } \Lambda(\phi), \text{ i.e., } t \in \Lambda^h, \nabla(t) = 0(h^\epsilon). \textrm{ Since the integration is extended only over } \Lambda^h, \int f^{(n)}(t) \Phi_h(t) dt = O(h^{\epsilon n}) \int \Phi_h(t) dt. \textrm{ The last integral obviously is } O(h^{-\delta}) \textrm{ for some } \delta \textrm{ (a careful estimate will be considered later) so choose } n > \delta/\epsilon.

The question of the structure of sets of type \( U^q \) is quite open. Let \( T \) be a closed set and \( |T| \) its Lebesgue measure. Obviously \( |T| = 0 \) is, in case \( q \leq 2 \) a sufficient, and in case \( q \geq 2 \) a necessary condition that \( T \) be of type \( U^q \). Exact criteria are available for \( q = 1 \) (\( T \) has empty interior), \( q = 2(|T| = 0) \), and \( q = \infty \) (\( T \) is empty). One would like to interpolate. The next theorem is a step in that direction which gives some content to Theorem 1. We consider \( r \)-tuple trigonometric series or integrals. \( \Lambda^h \) has the same meaning as in the paragraph above, and \( \dim T \) is the Hausdorff dimension of \( T \).

**Theorem 4.** \( \text{Alternative sufficient conditions that the closed set } T \text{ be of type } U^q, q \geq 2 \) are

1. \( |\Lambda^h| = o(h^{(1-2/q)}) \text{ for each compact subset } \Lambda \text{ of } T \),
2. \( \dim T < 2r/q \), \text{ with the proviso, if } r > 2 \text{, that } q \leq 2r/(r-2).

We shall give the proof of (i) only for ordinary trigonometric integrals. It suffices to show that if \( \phi \in L^q \) is a bounded continuous function with compact spectrum \( \Lambda(\phi) \subset T \) then \( \phi = 0 \). This will be true if, in the previous notation, \( \int |\Phi_h(t)| dt = o(1) \) as \( h \to 0 \). Using the Schwarz inequality, \( \{ \int |\Phi_h(t)| dt \}^2 \leq |\Lambda^h| \cdot \int |\Phi_h(t)|^2 dt \). Next we employ the Parseval relation and the Hölder inequality.

\[
\int |\Phi_h(t)|^2 dt = \int k(hx) \phi(x)^2 dx \\
\leq \left\{ \int k(hx)^{2q/(q-2)} dx \right\}^{1-2/q} \cdot \left\{ \int \phi(x)^q dx \right\}^{2/q} \\
= O(h^{1+2/q}) \cdot O(1).
\]

Combining the estimates, \( \{ \int |\Phi_h(t)| dt \}^2 = |\Lambda^h| \cdot O(h^{1+2/q}) = o(1) \) since \( |\Lambda^h| = o(h^{1-2/q}) \) by hypothesis. (ii) was proved by Beurling [1] for \( r = 1 \) and extended by Deny [2, pp. 144-145].

The conditions of Theorem 4 are clearly unnecessary since an ordinary set of uniqueness is of type \( U^q \) for every \( q, 1 \leq q < \infty \). However the estimates cannot be improved.
In conclusion we mention one amusing problem for $r$-dimensional Euclidean space. Suppose $f \in L^p$ and vanishes outside a compact set. Then $\hat{f}$ is an entire function of exponential type. For $r=1$, the translates of $f$ span $L^p$ for all $p$, $1 < p < \infty$, since $Z(f)$ is countable. However consideration of a few Bessel functions leads to the conclusion that for $r>1$ the theorem is certainly false unless $p \geq 2r/(r+1)$. Is this a sufficient condition? Posing the problem otherwise, for what $q$ is the set of real zeros of an entire function of exponential type in $r$-variables necessarily of type $U^q$?

**Bibliography**


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