ON SQUARE ROOTS OF NORMAL OPERATORS

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1. All operators in this paper are bounded (linear, everywhere defined) transformations on a Hilbert space of elements $x$. An arbitrary operator $A$ will be called a square root of a normal operator $N$ if

\[ A^2 = N. \]

It is clear that if $N$ possesses the spectral resolution $N = \int z dK(z)$, then any operator of the form $A = \int z^{1/2} dK(z)$, where, for the value of $z^{1/2}$, the choice of the branch of the function may depend on $z$, is a solution of (1). Moreover, all such operators are even normal.

Of course, equation (1) may have other, nonnormal, solutions $A$. The object of this note is to point out a simple condition to be satisfied by a square root $A$ guaranteeing that it be normal. This criterion will involve the (closed, convex) set $W = W_A$ consisting of the closure of the set of values $(Ax, x)$ where $||x|| = 1$. (Cf. also [2] wherein is discussed a connection between commutators and the set $W$.)

The following theorem will be proved:

(I) Let $N$ be a fixed normal operator and let $A$ denote an arbitrary solution of (1). Suppose that there exists a line $L$ in the complex plane passing through the origin and lying entirely on one side of (and possible lying all, or partly, in) the set $W_A$. Then $A$ is necessarily normal.

It is easy to see that the hypothesis of (I) concerning the line $L$ is surely satisfied if $W$ is a single point or a straight line segment. In this case, $A$ is even the sum of multiples of a self-adjoint operator and the unit operator $I$. (In fact, there exists some angle $\theta$ and some complex number $z$ such that the set $W$ belonging to $e^{i\theta}A + zI$ is a point or a segment of the real axis, and hence $e^{i\theta}A + zI$ is self-adjoint.) In case the set $W$ is actually two-dimensional, the assumption amounts to supposing that $0$ is not in the interior of $W$, although it is allowed of course that $0$ be on the boundary.

2. Proof of (I). Clearly, one can choose an angle $\theta$ for which the operator $B = e^{i\theta}A$ satisfies $B + B^* \geq 0$. If $B = H + iJ$, where $H = (B + B^*)/2$ and $J = -(B - B^*)/2$ denote the self-adjoint real and imaginary parts of $B$, then

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\( B^2 = e^{2i\theta}A^2 = (H^2 - J^2) + i(HJ + JH). \)

Since \( B^2 \) is normal and obviously commutes with \( B \), it follows that \( B^2 \) also commutes with \( B^* \); [1]. Consequently \( B^2 \) commutes with each of the operators \( H \) and \( J \). A subtraction of the two relations obtained from (2) by a multiplication by \( H \) on the left and on the right, respectively, now implies \( R + iS = 0 \), where \( R = J^2H - HJ^2 \) and \( S = H^2J - JH^2 \). On taking adjoints, one obtains \( R - iS = 0 \). Therefore \( S = 0 \), that is \( H^2J = JH^2 \); hence, since \( H \geq 0 \), \( HJ = JH \). Consequently \( B \), hence \( A \), is normal and the proof of (I) is now complete.

3. The following is a corollary of (I) and its proof:

(II) Let \( N \) be a fixed self-adjoint operator and let \( A \) denote a solution of (1) for which either (a) \( \Re(Ax, x) \neq 0 \) or (b) \( \Im(Ax, x) \neq 0 \) holds for all \( x \). Then either \( A \) or \( iA \) is self-adjoint according as (a) or (b) holds.

It should be noted that the hypothesis of (II) implies that the line \( L \) of (I) can be chosen either as the imaginary axis or as the real axis according as (a) or (b) holds and that, moreover, no number \( (Ax, x) \), for \( \|x\| = 1 \), actually lies on \( L \) (although, of course, such numbers may cluster at a point of \( L \)).

In order to prove (II), note that the angle \( \theta \) occurring in the proof of (I) can now be chosen to be 0 or \( \pi \) in case (a) and \( \pi/2 \) or \( 3\pi/2 \) in case (b). Furthermore, \( (Hx, x) > 0 \) whenever \( \|x\| = 1 \), so that 0 is not in the point spectrum of \( H \). Since \( e^{2i\theta} \) is real, it follows from the relation (2) that \( HJ + JH = 0 \). This fact combined with the relation \( HJ - JH = 0 \) implies \( HJ = 0 \), hence \( J = 0 \). Thus \( B \) (\( = H \)) is self-adjoint and so \( A = e^{-i\theta}B \). In view of the choice of \( \theta \), the proof of (II) is now complete.

References


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