ON A CLASS OF LATTICE-ORDERED RINGS

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1. Introduction. It is the purpose of this paper to study a class of rings called F-rings. An F-ring $R$ is a $\sigma$-complete vector lattice (Birkhoff [2, p. 238]) which is, in addition, a commutative algebra with a unit 1, satisfying the conditions

$$1 \geq 0; \quad x \geq 0, y \geq 0 \Rightarrow xy \geq 0; \quad x \land 1 = 0 \Rightarrow x = 0.$$

Here $\land$ denotes, as usual, the lattice operation greatest lower bound, and $x, y$ are elements of $R$. A bounded F-ring is an F-ring $\overline{R}$ such that each $x \in \overline{R}$ satisfies

$$x \lor 0 + (-x) \lor 0 \leq \lambda \cdot 1$$

for some real number $\lambda$, the symbol $\lor$ denoting the lattice least upper bound.

Any ring $R$ is regular [10] if for each $x \in R$ there is an $x^0 \in R$ such that $xx^0x = x$. It is evident that every regular F-ring $R$ contains a maximal bounded sub-F-ring $\overline{R}$, the F-ring of all $x \in R$ satisfying equation (1.1). The relationship between a regular F-ring and its maximal bounded sub-F-ring is analogous to that between the ring of all continuous functions on a completely regular space $X$ and the ring of all bounded continuous functions on $X$. For example, it is shown in Theorem 3 that there is a one-to-one correspondence between the maximal ideals of $R$ and those of $\overline{R}$. (For the theory of rings of continuous functions, see [5] and [6].)

A maximal ideal $M$ of a ring $R$ is real [6] if the quotient ring $R - M$ is ring-isomorphic to the real field. An ideal $S$ of an F-ring $R$ is closed if $a_n \in S$, $n \geq 1$ and $\bigvee_{n=1}^\infty a_n \in R$ imply $\bigvee_{n=1}^\infty a_n \in S$. It is proved in Theorems 5 and 6 that the closed maximal ideals of a regular F-ring are real and that there is a one-to-one correspondence between the closed maximal ideals of a regular F-ring $R$ and the closed maximal ideals of $\overline{R}$, the maximal bounded sub-F-ring of $R$.

It is a direct corollary of some results of Nakano [9, pp. 39, 212] that a bounded F-ring is ring- and lattice-isomorphic to the ring of all continuous functions on a compact Hausdorff space. Therefore every bounded F-ring is a semisimple real Banach algebra. “Real” is used here in the classical sense, that is, a partially ordered ring $R$ is

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real provided every element \( a \geq \epsilon \) for some real \( \epsilon > 0 \) has an inverse in \( R \). (The notation \( \epsilon \) is used in place of \( \epsilon \cdot 1 \).) An \( F \)-ring is called an \( M \)-ring if the intersection of all its real closed maximal ideals is the 0-element. A nontrivial example of a regular \( M \)-ring is the ring of all continuous functions on a \( P \)-space \([4]\). If \( \Phi \) is an abstract set and \( \mathcal{G} \) is a \( \sigma \)-algebra of subsets of \( \Phi \), then a real function \( f(\xi) \) defined on \( \Phi \) is said to be measurable or \((\Phi, \mathcal{G})\)-measurable if for each real \( \lambda \) the set \( \{\xi \mid f(\xi) \leq \lambda\} \) belongs to \( \mathcal{G} \).

The main result of this paper (Theorem 7) is that any real \( M \)-ring \( B \) is ring- and lattice-isomorphic to \( B(\Omega, \mathcal{A}) \), an \( F \)-ring of \((\Omega, \mathcal{A})\)-measurable functions. Here \( \Omega \) designates the set of real closed maximal ideals of \( B \), and \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of \( \Omega \) isomorphic to the Boolean algebra \( I \) of idempotents of \( B \). In addition (corollary to Theorem 8), if \( B \) is a regular, then \( B(\Omega, \mathcal{A}) \) is the \( F \)-ring of all \((\Omega, \mathcal{A})\)-measurable functions.

2. Notation and properties of \( F \)-rings. In what follows, \( R \) always denotes a regular \( F \)-ring, \( \overline{R} \) denotes its maximal bounded sub-\( F \)-ring, and \( I \) denotes the set of idempotents of \( R \). Latin letters denote elements of rings, and Greek letters denote real numbers. Additional definitions are

\[
x^+ = x \lor 0, \quad x^- = (-x) \lor 0, \quad |x| = x^+ + x^-, \quad \bar{e}_x = \bigvee_{n=1}^{\infty} n|x| \land 1, \quad \text{and} \quad e_x = 1 - \bar{e}_x.
\]

The following properties of \( F \)-rings (Nakano \([9, \text{Chapters I and IV}]\)) are used throughout:

(N 1) For every \( b \in R \), \( a = V_{\lambda \in A} a_\lambda \) implies

\[
a \land b = V_{\lambda \in A} (a_\lambda \land b)
\]

and the dual statement, that is, the statement with \( \lor \)'s and \( \land \)'s replaced respectively by \( \land \)'s and \( \lor \)'s.

(N 2) If \( a = \Lambda_{\lambda \in A} a_\lambda \), then \( a + b = \Lambda_{\lambda \in A} (a_\lambda + b) \) and if \( a = V_{\lambda \in A} a_\lambda \), then \( a + b = V_{\lambda \in A} (a_\lambda + b) \) for all \( b \in R \).

(N 3) If \( a = \Lambda_{\lambda \in A} a_\lambda \), then \( ab = \Lambda_{\lambda \in A} a_\lambda b \), and if \( a = V_{\lambda \in A} a_\lambda \), then \( ab = V_{\lambda \in A} a_\lambda b \), for all \( b \geq 0 \).

(N 4) For all \( a \in R \), \( a^2 \geq 0 \).

(N 5) If \( a \geq 0 \) and \( b \geq 0 \), then \( a \land b = 0 \) is equivalent to \( ab = 0 \).

(N 6) \( R \) is archimedian, that is, for every non-negative element \( a \in R \), \( \bigwedge_{n=1}^{\infty} (1/n)a = 0 \).

From (N 5), an element \( e \in R \) is an idempotent if and only if \( e \land (1 - e) = 0 \). By the methods of \([8]\), \( I \) forms a Boolean algebra with
respect to the lattice operations of $R$. Since for $a, b \in I$, $ab \leq a \wedge b$, and by (N 3), $a \wedge b = (a \wedge b)^2 = (a \wedge b) \wedge ab$, it follows that $ab = a \wedge b$. In addition, a further application of (N 3) shows that $I$ is $\sigma$-complete.

From some results in [8, p. 530], it follows that (N 7) $\bar{e}_z$ is idempotent.

In the sequel, $(P)$ will denote the ideal of $R$ generated by the subset $P$ of $R$, and $(x)$ will denote the ideal generated by $x \in R$.

3. Regular $F$-rings. This section begins with a proof of a general result concerning regular rings.

**Theorem 1.** Let $A$ be a commutative ring with a unit $1$. $A$ is regular if and only if it has property

(a) For each $x \in A$ there exists an element $ax \in A$ such that, first, $a \cdot x = ax$, second, $xax = 0$, third, $x + ax$ has an inverse.

If $x^0$ is the element postulated in the definition of regularity, then $a_x = 1 - xx^0$.

**Proof.** If $A$ has property (a), then there exists $y \in A$ such that $y(x + ax) = 1$. Therefore $x = x \cdot 1 = xy(x + ax) = xyx$; that is, $A$ is regular.

If $A$ is regular, then there is an element $x^0 \in A$ such that $xx^0x = x$. It then follows, first, that $xx^0$ is an idempotent of $A$, second, that $x(1 - xx^0) = 0$, third, that $1 - xx^0$ is idempotent, and, fourth, that

$$[x + (1 - xx^0)] [x(x^0)^2 - xx^0 + 1] = 1.$$  

Thus with $a_x = 1 - xx^0, A$ has property (a).

**Theorem 2.** If $x^0 \in R$ has the property $xx^0x = x$, then $xx^0 = \bar{e}_x$ and $1 - xx^0 = e_x$.

**Proof.** This consists in showing that $x\bar{e}_x = x$ and that $\bar{e}_x \in (x)$. From $xx \bar{e}_x = x$, it follows that $(\bar{e}_x x)x^0 = xx^0$, and from $\bar{e}_x \in (x)$ it follows that $\bar{e}_x = xy$ for some $y \in A$, from which we deduce $\bar{e}_x xx^0 = xyxx^0 = xy$. Thus $\bar{e}_x = xx^0$.

To show that $\bar{e}_x x = x$, note that

$$(2.1) \quad x^+ \wedge \bar{e}_x = \bigvee_{n=1}^{\infty} x^+ \wedge n \mid x \mid \wedge 1 = x^+ \wedge 1$$

and that, by the same reasoning,

$$(2.2) \quad x^- \wedge \bar{e}_x = x^- \wedge 1.$$  

Since by (N 7) $\bar{e}_x$ is idempotent, equations (2.1) and (2.2) imply

$$0 = x^+ \wedge \bar{e}_x \wedge (1 - \bar{e}_x) = x^+ \wedge 1 \wedge (1 - \bar{e}_x) = x^+ \wedge (1 - \bar{e}_x)$$

and, by a similar line of reasoning, it follows that
From (N 5), (2.3), and (2.4), it follows that \( x^+ \cdot \bar{e}_z = x^+ \) and \( x^- \cdot \bar{e}_z = x^- \), and hence \( x \bar{e}_z = x \).

Finally we consider the proposition \( \bar{e}_z \in (x) \). Since \( R \) is regular, the principal ideals \((x), (|x|), (xx^0)\) are all equal. By (N 4), \( xx^0 \geq 0 \) and 1 \(- xx^0 \geq 0 \), and by (N 5), \( |x| \cdot (1 - xx^0) = 0 \). Hence \((x)\) is closed under countable sup's (see (N 3)) and \((x)\) is an \( \ell \)-ideal in the sense of Birkhoff [2, p. 222], that is, if \( |y| \leq |x| \), then \( y \in (x) \). Since \( n |x| \in (x) \), and since \((x)\) is an \( \ell \)-ideal, it follows that \( n |x| \wedge 1 \in (x) \) and \( \bigvee_{n=1}^{\infty} n |x| \wedge 1 = \bar{e}_z \in (x) \).

**Corollary.** \( R \) is real.

**Proof.** If \( x \geq \varepsilon > 0 \), then \( n |x| \wedge 1 \geq n \varepsilon \wedge 1 \) and hence \( \bar{e}_z = 1 \). Since \( xx^0 = \bar{e}_z \), it follows (Theorem 1) that \( e_z = 1 - xx^0 = 0 \) and that \( x^{-1} \) exists.

**Corollary.** If \( B \) is a partially ordered subring of \( R \) containing \( \overline{R} \), then \( B \) is real.

**Proof.** If \( x \in B \) and \( x \geq \varepsilon > 0 \), then \( (x + e_z)^{-1} = x^{-1} \in R \) and \( 0 < x^{-1} \leq 1/\varepsilon \). Therefore \( x^{-1} \in \overline{R} \subset B \).

4. **Maximal ideals of regular \( F \)-rings.** This section is devoted to a discussion of the relationship between the maximal (ring) ideals of \( R \) and those of \( \overline{R} \).

**Theorem 3.** There is a one-to-one correspondence \( (M \rightarrow \phi(M)) \) between the maximal ideals of \( R \) and those of \( \overline{R} \).

**Proof.** Definition of \( \phi \): Since \( \overline{R} \) is ring- and lattice-isomorphic to the ring of continuous functions on a compact Hausdorff space, a result of Gillman and Henriksen [4, Theorem 3.3] shows that each prime ideal \( \overline{P} \) of \( \overline{R} \) is contained in a unique maximal ideal of \( R \). If \( M \) is a maximal ideal of \( R \), then \( \overline{R} \cap M \) is a prime ideal of \( \overline{R} \) and is contained in a unique maximal ideal \( \phi(M) \) of \( \overline{R} \). The mapping \( \phi \) is then a single valued mapping of the maximal ideals of \( R \) into the maximal ideals of \( \overline{R} \).

To show \( \phi \) is a mapping onto the maximal ideals of \( \overline{R} \), suppose \( \overline{M} \) is a maximal ideal of \( \overline{R} \). Then \( \overline{M} \cap I \) is a prime ideal of the Boolean algebra \( I \). Since \( R \) is regular and commutative, a result of Morrison [7] states that \( (\overline{M} \cap I) \) is a maximal ideal of \( R \). The prime ideal \( (\overline{M} \cap I) \cap \overline{R} \) of \( \overline{R} \) contains \( \overline{M} \cap I \), so it must be contained in \( \overline{M} = \phi ([\overline{M} \cap I]) \).

To establish that \( \phi \) is biunique, it is first necessary to show that if
P is a prime ideal of \( \mathcal{R} \), then either \( (P) \) is equal to \( R \) or it is a maximal ideal. Indeed, suppose \( (P) \) is different from \( R \). Let \( a \) be an arbitrary element of \( R \) not in \( (P) \). The ideal \( (a, (P)) \), generated by \( a \) and \( (P) \), contains \( a + e_a \). For if \( a \notin (P) \), then \( e_a \notin (P) \) because \( a e_a = a \). Since \( e_a \) does not belong to \( P \) either, it follows that \( 1 - e_a = e_a \in P \). The regularity of \( R \) implies that \( a + e_a \) possesses an inverse, so \((a, (P)) \) equals \( R \) and \( (P) \) is a maximal ideal. If \( M_1 \) and \( M_2 \) are maximal ideals of \( R \) where \( M_1 \cap \mathcal{R} \) and \( M_2 \cap \mathcal{R} \) are both subsets of the same maximal ideal \( M \) of \( \mathcal{R} \), then \( M \cap I = M_1 \cap \mathcal{R} \cap I = M_2 \cap \mathcal{R} \cap I \) is a prime ideal of \( I \). Therefore from [7] it follows that \( M_1 = M_2 = (M \cap I) \).

As in the case of the ring of all continuous functions on a completely regular space (see [6] for example), there is no guarantee that all maximal ideals of \( R \) are real. Real maximal ideals are characterized by the following theorem.

**Theorem 4.** A necessary and sufficient condition for a maximal ideal \( M \) of \( R \) to be real is that \( M \cap \mathcal{R} \) be a maximal ideal of \( \mathcal{R} \).

**Proof.** If \( M \) is a real maximal ideal of \( R \), then

\[
(4.1) \quad R - M \supseteq [\mathcal{R} + M] - M \cong \mathcal{R} - M \cap \mathcal{R}
\]

by the second homomorphism theorem for rings. The left-hand member is isomorphic to the real field and the right-hand member contains a field isomorphic to the real field. Therefore \( \mathcal{R} - M \cap \mathcal{R} \) is isomorphic to the real field; hence \( M \cap \mathcal{R} \) is a maximal ideal.

Let \( M \) be a maximal ideal of \( R \). If \( M \cap \mathcal{R} \) is a maximal ideal of \( \mathcal{R} \), then formula (4.1) implies that \([\mathcal{R} + M] - M \) is isomorphic to the real field.

In order to finish the proof, it suffices to show that \( \mathcal{R} + M = R \). The following inequality can be proved for each pair of real numbers \( \lambda < \mu \) and each \( x \in R \), using the fact that \( \{ e_{(x - \lambda)}^+ \} \) is a spectral decomposition of 1 relative to \( x \) [2, p. 251] and that \( xy = 0 \) implies \( e_x y = y \). If \( e_x (\lambda, \mu) \) stands for \( e_{(x, \lambda)}^+ - e_{(x, -\lambda)}^+ \), then the inequality can be expressed as follows

\[
(4.2) \quad \lambda e_x (\lambda, \mu) \leq x e_x (\lambda, \mu) \leq \mu e_x (\lambda, \mu).
\]

Therefore \( x e_x (\lambda, \mu) \in \mathcal{R} \).

Let \( x \) be an element of \( R \) not in either \( M \) or \( \mathcal{R} \). Suppose \( x e_x (\lambda, \mu) \in M \cap \mathcal{R} \) for all \( \lambda, \mu \quad (\lambda < \mu) \). Then \( e_x (\lambda, \mu) \in M \cap \mathcal{R} \) for all such \( \lambda, \mu \) and in addition

\[
(4.3) \quad a = \sum_{N=1}^{\infty} \frac{1}{N^2} e_x (N - 1, N) + \sum_{N=1}^{\infty} \frac{1}{N^2} e_x (-N, -N + 1)
\]
belongs to $M \cap \overline{R}$ because the maximal ideals of the real Banach algebra $\overline{R}$ are norm-closed.

Since \( \{ e_{(x-\lambda)^+} \} \) is a spectral decomposition,

\[
e^x(\lambda, \mu) \wedge e^x(\sigma, \tau) = e^x(\lambda, \mu) \cdot e^x(\sigma, \tau) = 0
\]

if the closed intervals $[\lambda, \mu]$ and $[\sigma, \tau]$ have no more than one point in common. Therefore equation (4.3) can be replaced [9, Theorem 5.15] by

\[
(4.4) \quad a = \bigvee_{N=1}^{\infty} \frac{e^x(N - 1, N)}{N^2} \bigvee_{M=1}^{\infty} \frac{e^x(-M, -M + 1)}{M^2}.
\]

Now

\[
\tilde{e}_a = \bigvee_{n=1}^{\infty} n a \wedge 1
\]

\[
= \bigvee_{n=1}^{\infty} \left[ \bigvee_{N=1}^{\infty} \frac{n}{N^2} e^x(N - 1, N) \bigvee_{M=1}^{\infty} \frac{n}{M^2} e^x(-M, -M + 1) \right] \wedge 1
\]

\[
= \bigvee_{n,N,M} \left[ \frac{n}{N^2 M^2} e^x(N - 1, N) \bigvee_{M=1}^{\infty} \frac{n}{M^2} e^x(-M, -M + 1) \right] \wedge 1
\]

and because $e = e^2$ implies $n e \wedge 1 = e$ for all $n \geq 1$, it follows that

\[
\tilde{e}_a = \bigvee_{N,M} \left[ e^x(N - 1, N) \bigvee e^x(-M, -M + 1) \right]
\]

\[
= 1.
\]

Therefore $e_a = 0$ and $a^{-1}$ belongs to $R$. Since this is impossible because $a \in M$, the supposition that $e^x(\lambda, \mu) \in M$ for all pairs $(\lambda, \mu)$ is incorrect.

Let $\lambda$ and $\mu$ be numbers such that $e^x(\lambda, \mu) \in M$. Then $1 - e^x(\lambda, \mu) \in M \cap \overline{R}$. For any $x \in R$, the element $xe^x(\lambda, \mu)$ belongs to $\overline{R}$, and $x - xe^x(\lambda, \mu)$ belongs to $M$. Thus $R = \overline{R} + M$, which concludes the proof.

Closed maximal ideals figure importantly in what follows. We therefore conclude this section with a few facts about them.

**Theorem 5.** The closed maximal ideals of $R$ are real.

**Proof.** In order to show that a closed maximal ideal $M$ is real, it suffices to show that $R - M$ is the real field. In the course of the proof of Theorem 2, it was shown that $(x)$, and hence any maximal ideal $M$ of $R$, is an $l$-ideal. Thus the quotient space $R - M$ is an $l$-group.
[2, pp. 214, 222]. If \( a(M) \) stands for the image of \( a \in R \) under the natural homomorphism of \( R \) onto \( R - M \), then \( a^+(M) \cdot a^-(M) = 0 \). Since \( R - M \) is a field, either \( a^+(M) = 0 \) or \( a^-(M) = 0 \); hence \( R - M \) is simply ordered. We know that \( R - M \) is an ordered field because the following statement is a trivial consequence of the definition of order in \( R - M \): \( a(M) \geq 0 \) and \( b(M) \geq 0 \) imply \( a(M) \cdot b(M) = 0 \).

That \( R - M \) is a \( \sigma \)-complete vector lattice follows from the hypothesis that \( M \) is a closed maximal ideal. Under these circumstances, \( R - M \) is archimedian (N 6) and hence it is isomorphic to the real field [2, Ex. 2, p. 229].

**Theorem 6.** There is a one-to-one correspondence between the closed maximal ideals of \( R \) and those of \( \overline{R} \).

**Proof.** If \( M \) is a closed ideal of \( R \), then \( \phi(M) = M \cap \overline{R} \) is a maximal ideal of \( \overline{R} \) (Theorems 4 and 5). The ideal \( M \cap \overline{R} \) is also easily seen to be closed.

If, on the other hand, \( \overline{M} \) is a closed maximal ideal of \( \overline{R} \), then \( \overline{M} \cap I \) is a \( \sigma \)-prime ideal of the Boolean algebra \( I \), that is, \( \overline{M} \cap I \) is a prime ideal of \( I \) satisfying the added property: if \( a_n \in \overline{M} \cap I \) for all integers \( n \geq 1 \) and \( a = \bigvee_{n=1}^\infty a_n \in I \), then \( a \in \overline{M} \cap I \). The ideal \( (\overline{M} \cap I) \) of \( R \) is, by Morrison's Theorem [7], a maximal ideal. In addition, \( (\overline{M} \cap I) \cap \overline{R} = \overline{M} \).

To finish the proof it suffices to show that \( (\overline{M} \cap I) \) is a closed ideal of \( R \). Suppose \( x_n \in (\overline{M} \cap I) \) for \( n \geq 1 \) and suppose \( x = \bigvee_{n=1}^\infty x_n \) belongs to \( R \). Then \( \varepsilon x_n \in \overline{M} \cap I \) for \( n \geq 1 \) and because \( \varepsilon x_n^+ \leq \varepsilon x_n^-(\text{easily verifiable}) \),

\[
\bigvee_{m=1}^\infty \varepsilon x_m^+ = \bigvee_{m=1}^\infty \bigvee_{n=1}^\infty n x_m^+ \wedge 1 = \bigvee_{n=1}^\infty n \left( \bigvee_{m=1}^\infty x_m^+ \right) \wedge 1.
\]

Therefore \( \varepsilon x^+ \in \overline{M} \cap I \), and also \( x^+ \in (\overline{M} \cap I) \). That \( x^- \) for \( n \geq 1 \) and \( x^- = \bigwedge_{n=1}^\infty x_n^- \) all belong to \( (\overline{M} \cap I) \) follows because \( (\overline{M} \cap I) \) is an \( l \)-ideal. Thus \( x^+ - x^- = x \in (\overline{M} \cap I) \); hence \( (\overline{M} \cap I) \) is closed.

5. **Representation theorems for certain \( F \)-rings.** It is clear that the ring of all \((\Phi, \mathcal{Q})\)-measurable functions is a regular \( M \)-ring. Indeed, each point \( \xi \in \Phi \) corresponds to the closed maximal ideal of all functions vanishing at \( \xi \), so the only function common to all closed maximal ideals is the zero function. The ring of all \((\Phi, \mathcal{Q})\)-measurable functions contains every \( F \)-ring of \((\Phi, \mathcal{Q})\)-measurable functions. In this section it is shown that, conversely, every regular \( M \)-ring is ring-and lattice-isomorphic to the \( M \)-ring of all \((\Phi, \mathcal{Q})\)-measurable functions for a certain well defined pair \((\Phi, \mathcal{Q})\).

In the remainder of §5, \( B \) is used to denote a real \( M \)-ring, \( \Omega \) to
denote the set of all real closed maximal ideals $M$ of $B$, and $J$ to denote the Boolean algebra of idempotents of $B$. If $M \in \Omega$, then $x(M)$ stands for the image of $x$ under the natural homomorphism of $B$ onto $B - M$. The symbol $x(\cdot)$ represents the real valued function defined on $\Omega$ which takes the value $x(M)$ at the point $M \in \Omega$. For each $e \in J$ consider the subset $U(e) = \{ M \mid e(M) = 1 \}$. $\mathfrak{A}$ is used to denote the collection of all such subsets.

**Lemma.** $\mathfrak{A}$ is a $\sigma$-algebra and is isomorphic to $J$.

**Proof.** If $M$ is a closed ideal, then $M \cap J$ is a $\sigma$-prime ideal of $J$ and if $\bigcap_\omega M = 0$, then $\bigcap_\omega M \cap J = 0$ also. By a result of Sikorski [11, Theorem 1.3], $\mathfrak{A}$ is a $\sigma$-algebra and is isomorphic to the $\sigma$-complete Boolean algebra $J$.

Now we may use $R(\Omega, \mathfrak{A})$ to denote the $M$-ring of all $(\Omega, \mathfrak{A})$-measurable functions and $\overline{R}(\Omega, \mathfrak{A})$ to denote the $M$-ring of all bounded $(\Omega, \mathfrak{A})$-measurable functions.

**Theorem 7.** $B$ is ring- and lattice-isomorphic to $B(\Omega, \mathfrak{A})$, an $M$-ring of $(\Omega, \mathfrak{A})$-measurable functions.

**Proof.** First, by the standard (Gelfand [3]) argument, $B$ can be shown to be ring-isomorphic to a ring $B(\Omega)$ of real valued functions defined on $\Omega$.

The mapping $x \rightarrow x(M)$ of $B$ onto $B - M$ preserves order. Indeed, suppose $x \geq 0$ and $x(M) \leq 0$. Then $x - x(M) \geq -x(M) \geq 0$, and, because $B$ is real, $x - x(M)$ has an inverse. However, $x - x(M)$ belongs to $M$. Therefore $x \geq 0$ is a sufficient condition for $x(M) \geq 0$; hence the mapping preserves order.

Define $x(\cdot) \geq y(\cdot)$ if $x(M) \geq y(M)$ for each $M \in \Omega$. This definition induces a partial order on $B(\Omega)$, and with $B(\Omega)$ thus ordered, the isomorphism mentioned in the first paragraph of this proof preserves order. Necessarily the lattice structure is preserved as well; hence $B$ is ring- and lattice-isomorphic to $B(\Omega)$.

Finally, each $x(\cdot) \in B(\Omega)$ is a $(\Omega, \mathfrak{A})$-measurable function. Indeed, each $M \in \Omega$ is an $l$-ideal as well as a real closed maximal ideal. (To see that this is so, look at the image of $M$ in $B(\Omega)$.) From this, it follows that $x \in M$ if and only if $\bar{e}_x \in M$. Therefore

$$U[e_{(x-\lambda)^+}] = \{ M \mid e_{(x-\lambda)^+}(M) = 1 \}$$

$$= \{ M \mid (x - \lambda)^+(M) = 0 \}$$

$$= \{ M \mid x(M) \leq \lambda \}$$

belongs to $\mathfrak{A}$ for each $\lambda$. Thus the symbol $B(\Omega)$ may be meaningfully replaced by the symbol $B(\Omega, \mathfrak{A})$ and the theorem is proved.
It should be remarked at this point that Theorem 7 is still valid when $\Omega$ is replaced by any subset $\Omega^*$ where the intersection of all ideals in $\Omega^*$ is the zero ideal. $\mathcal{A}$ is defined in the same manner with respect to $\Omega^*$ as it was with respect to $\Omega$.

The following theorem is closely related to Theorem 7.

**Theorem 7A.** The following statements are equivalent. (i) $R = \overline{R}$. (ii) $R$ is the $F$-ring of all ordered $n$-tuples of real numbers for some fixed integer $n$. (iii) All maximal ideals of $R$ are closed.

**Proof.** (i) implies (ii). Since $R = \overline{R}$ is a regular real semisimple commutative Banach algebra, it is finite dimensional [1, Theorem 3.5] and it has a representation as a ring of functions. The result follows because the number of maximal ideals of $R$ is then finite.

(ii) implies (i) and (ii) implies (iii) are trivial.

(iii) implies (ii). If all maximal ideals of $R$ are closed, then the mapping $M \to M \cap \overline{R}$ is a one-to-one correspondence between the maximal ideals of $R$ and those of $\overline{R}$ (Theorems 4, 5, and 6), and in addition each maximal ideal of $\overline{R}$ is closed (Theorem 6).

Since every maximal ideal $M$ of $R$ is closed, it follows easily that $x \in M$ if and only if $x + e_x \in M$. Hence if $x \in \overline{R}$, $x + e_x$ belongs to no maximal ideal of $\overline{R}$. Therefore $x + e_x$ has an inverse in $\overline{R}$; so $\overline{R}$ is regular (Theorem 1). By an argument similar to that used in the first paragraph of the proof, $\overline{R}$ is finite dimensional. From the one-to-one correspondence $M \to M \cap \overline{R}$, we deduce that $R$ is also finite dimensional, and therefore, by Theorem 7, Statement (ii) follows.

**Definition.** An algebra $A$ of $(\Phi, \mathcal{X})$-measurable functions is $\sigma$-convex provided that $y(\cdot) \in A$ if $y(\cdot)$ is $(\Phi, \mathcal{X})$-measurable and $0 \leq y(\cdot) \leq x(\cdot)$ where $x(\cdot)$ belongs to $A$.

Such a $\sigma$-convex algebra is necessarily a real $M$-ring; the following theorem is a converse to this statement.

**Theorem 8.** $B(\Omega, \mathcal{A})$ (defined in Theorem 7) is a $\sigma$-convex algebra.

**Proof.** In order to show $B(\Omega, \mathcal{A})$ is $\sigma$-convex, it is first necessary to show $\overline{B}(\Omega, \mathcal{A}) \subseteq B(\Omega, \mathcal{A})$. It should be noted that since a lattice isomorphism preserves sup's and inf's whenever they occur, the definition of order in $B(\Omega, \mathcal{A})$ implies that the sup in $B(\Omega, \mathcal{A})$ is the pointwise sup. Since $B(\Omega, \mathcal{A})$ contains all simple functions (finite linear combinations of characteristic functions of sets in $\mathcal{A}$), and since every bounded measurable function is the pointwise sup of a countable set of simple functions, it follows from the conditional $\sigma$-completeness of $B(\Omega, \mathcal{A})$ that $\overline{B}(\Omega, \mathcal{A})$ is a sub-$M$-ring of $B(\Omega, \mathcal{A})$.

To finish the proof, suppose that $y(\cdot) \geq 0$ is a $(\Omega, \mathcal{A})$-measurable function and that $y(\cdot) \leq x(\cdot) \in B(\Omega, \mathcal{A})$. Let $y_n(\cdot) = y(\cdot) \chi_M \{ M \mid y(M)$
Each element $y_n(\cdot)$ is bounded; hence it belongs to $B(\Omega, \mathcal{A})$. In addition, $y_n(\cdot) \leq x(\cdot)$ for each $n$. Therefore since $B(\Omega, \mathcal{A})$ is an $M$-ring, $y(\cdot) = \bigvee_{n=1}^{\infty} y_n(\cdot)$ also belongs to $B(\Omega, \mathcal{A})$.

**Corollary.** If $B$ is regular, then $B(\Omega, \mathcal{A}) = R(\Omega, \mathcal{A})$.

**Proof.** Suppose $0 \leq x(\cdot)$ is an $(\Omega, \mathcal{A})$-measurable function. Then $x(\cdot) + 1$ is also $(\Omega, \mathcal{A})$-measurable and $y(\cdot) = (x(\cdot) + 1)^{-1}$ is both measurable and bounded. By Theorem 8, $y(\cdot) \in B(\Omega, \mathcal{A})$, and since $y(M) \neq 0$ for all $M \in \Omega$, it follows that the characteristic function $\chi \{ M \mid y(M) = 0 \} = 0$ and if $y$ is the isomorphic copy in $B$ of $y(\cdot)$, then the isomorphic copy $e_y$ of the characteristic function $\chi \{ M \mid y(M) = 0 \}$ is zero. Therefore, $y^{-1}$ exists in $B$, and $y^{-1}(\cdot) = x(\cdot) + 1 \in B(\Omega, \mathcal{A})$. Thus $x(\cdot) \in B(\Omega, \mathcal{A})$, and the corollary follows from this.

An $F$-ring $A$ is atomic if its Boolean algebra of idempotents is an atomic Boolean algebra. We conclude with two theorems about atomic $F$-rings.

**Theorem 9.** If $R$ is atomic, then $R$ is an $M$-ring.

**Proof.** In the course of the proof of Theorem 6, it was shown that there is a one-to-one correspondence $\mathcal{M} \rightarrow \mathcal{M}(\Omega)$ between the set of closed maximal ideals of $R$ and the set of $\sigma$-prime ideals of $I$. Sikorski [11, Theorem 1.8] has shown that if a $\sigma$-complete Boolean algebra is atomic, then the intersection of its $\sigma$-prime ideals is zero. Since $x \in M$ if and only if $\varepsilon_x \in M \cap I$, it follows that
\[
\bigcap_{M \in \Omega} M = \bigcap_{M \in \Omega} M \cap I = 0;
\]
hence $R$ is an $M$-ring.

An $F$-ring $A$ is complete if any arbitrary set of elements of $A$, bounded above by an element of $A$, has a least upper bound.

**Theorem 10.** If $R$ is both complete and atomic, then it is a direct sum of real fields.

**Proof.** To show that $R$ is a direct sum of real fields, it suffices to show that $R$ is isomorphic to the ring of all real-valued functions on some space $\Omega^*$. Let $\Omega^*$ be the set of maximal ideals $M_a = \{ x \mid xa = 0 \}$ where $a$ is an atom of $I$. Each ideal $M_a$ is closed and since $\cap_{a \in I} M_a \cap I = 0$, we deduce that $\cap_{a \in I} M_a = 0$ also (see proof of Theorem 10).

By the remark following the proof of Theorem 7, $R$ is isomorphic to $R(\Omega^*, \mathcal{A})$ ($\mathcal{A}$ is defined relative to $I$ and, by Lemma, is isomorphic to $I$). The complete atomic character of $R$ insures that $\mathcal{A}$ is the Boolean algebra of all subsets of $\Omega^*$. Hence $R(\Omega^*, \mathcal{A})$ is the $M$-ring of all real functions on $\Omega^*$.  

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