NORMAL SUBGROUPS OF MONOMIAL GROUPS

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1. Introduction. Let $U$ be a set, $o(U) = B = \aleph_u$, $u \geq 0$, where $o(U)$ means the number of elements of $U$. Let $H$ be a fixed group. A monomial substitution $y$ is a transformation that maps every $x$ of $U$ in a one-to-one fashion into an $x$ of $U$ multiplied by an element $h_x$ of $H$. Multiplication of substitutions means successive applications. The set of all monomial substitutions forms a group $\Sigma$. Ore [1] has studied this group for finite $U$, and some of his results have been generalized to general $U$ in [3].

The normal subgroups for one subgroup of $\Sigma$ have been determined [3]. This paper extends those results to the extent of determining the structure of all normal subgroups of various subgroups of $\Sigma$ in a rather general case. These results are stated in Theorems 1, 2, 3, 4, 5, 6.

2. Definitions. Let $d$ be the cardinal of the integers. Let $B$ be an infinite cardinal; $B^+$, the successor of $B$; $U$, a set such that $o(U) = B$; and $C$ such that $d < C \leq B^+$. Let $H$ be a fixed group and $e$ the identity of $H$. Denote by $\Sigma = \Sigma(H, B, d, C)$ the monomial group of $U$ over $H$ whose elements are of the form

$$y = \left( \cdots, x_e, \cdots \right) \left( \cdots, h_x x_i, \cdots \right)$$

where only a finite number of the $h_x$ are not $e$ and the number of $x$ not mapped into themselves is less than $C$. Any element of $\Sigma$ may be written in the form

$$y = \left( \cdots, x_e, \cdots \right) \left( \cdots, h_x x_i, \cdots \right) \left( \cdots, e x_i, \cdots \right)$$

or $y = vs$ where $v$ sends every $x$ into itself, every $h$ of $s$ is $e$. Elements of the form of

$$v = \left( \cdots, x_e, \cdots \right) = \{ \cdots, h_x, \cdots \}$$

are multiplications and all such elements form a normal subgroup,

Presented to the Society, December 28, 1956; received by the editors March 18, 1956 and, in revised form, December 22, 1956.

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the basis group \( V(B, d) = V \) of \( \Sigma \). The \( h_\epsilon \) of \( y \) are called the factors of \( y \). Elements of the form of \( s \) are permutations and all such elements form a subgroup, the permutation group, \( S(B, C) = S \) of \( \Sigma(H; B, d, C) \). Cycles of \( s \) will also be written as \((x_1, \ldots, x_n)\) and \((\ldots, x_{-1}, x_0, x_1, \ldots)\). Baer [2] has shown that the normal subgroups of \( S(B, C) \) are the alternating group, \( A = A(B, d) \), and \( S(B, D) \) where \( d \leq D \leq C \). Let \( E \) be the identity of \( \Sigma \), \( I \) the identity of \( S \).

3. Normal subgroups of \( \Sigma \) contained in the basis group.

**Theorem 1.** Let \( G_1 \subseteq G \) be normal subgroups of \( H \) such that \( G/G_1 \) belongs to the center of \( H/G_1 \). Let \( N \) be the set of multiplications with a finite number of different from \( e \) factors subject to the conditions that the factors are in \( G \) and their product, in any order, belongs to \( G_1 \). \( N \) is a normal subgroup of \( \Sigma \) contained in the basis group. Conversely, let \( N \) be a normal subgroup of \( \Sigma \) contained in the basis group. For every \( v \) of \( N \) form all possible products of its nonidentity factors. This set \( G_1 \) is a normal subgroup of \( H \). The set \( G \) consisting of every element of \( h \) that occurs as a factor in any \( v \) of \( N \) is also a normal subgroup of \( H \). Further, \( G/G_1 \) belongs to the center of \( H/G_1 \).

**Proof.** It is clear that \( N \) is a subgroup. Let \( y_1 = v_1 s_1 \) be any element of \( \Sigma \). Then \( y_1 v y_1^{-1} = v_1 s_1 v s_1^{-1} v_1^{-1} = v_1 v_2 v_1^{-1} \) where \( v_2 \) belongs to \( N \). Since \( G/G_1 \) is in the center of \( H/G_1 \) the product of the factors of \( v_1 v_2 v_1^{-1} \) is in \( G_1 \) and \( N \) is normal in \( \Sigma \).

If \( N \) is normal in \( \Sigma \) then an argument similar to that in [3, pp. 204—207] shows it has the structure described in the theorem.

4. Normal subgroups of \( \Sigma \) not contained in the basis group. The problem of finding all normal subgroups of \( \Sigma \) will be concluded by finding those normal subgroups not in \( V \).

**Lemma 1.** Let \( M \) be normal in \( \Sigma \), \( V \upharpoonright M \). Then \( N = M \cap V \) is normal in \( \Sigma \) and \( G = H \), i.e., the factors in any fixed position run through \( H \) and \( H/G_1 \) is Abelian.

**Proof.** Choose \( y = vs \epsilon M \) such that \( s \neq I \). Let \( v = \{k_1, k_2, \ldots\} \) be arbitrary in \( V \). Then the multiplication \( y^{-1} v^{-1} y v \) is in \( M \). Let \( s \) send \( x_1 \) into \( x_i \) with \( \epsilon \neq i \). The factor occurring in the position occupied by \( x_{i_\epsilon} \) is \( h^{-1} k^{-1}_\epsilon h k_{i_\epsilon} \). Since \( v \) is arbitrary and \( i_\epsilon \neq \epsilon \) we may choose the factors \( k^{-1}_\epsilon \), \( k_{i_\epsilon} \) in such a way that the factor above is arbitrary in \( H \).

**Lemma 2.** Let \( M \) be normal in \( \Sigma \), \( V \upharpoonright M \). Then \( P = M \cap S \) is normal in \( S \) and \( P \neq E \).

**Proof.** Let \( y \) be an element of \( M \) and \( y = vs \) where \( s \neq I \). Since \( y \)
has only a finite number of different from $e$ factors, $M$ must contain a $y'$ conjugate to $y$ where the finite cycles of $y$ have been written in normal form, [1, p. 20]. If $y$ contains an infinite cycle then $y'$ contains an infinite cycle in the form,

$$\left( \cdots, x_{-1}, x_0, x_1, \cdots \right).$$

If $y'$ contains a finite cycle then $[1$, pp. 35–36] $M$ contains a permutation. If $y'$ contains only infinite cycles let $s_1 = (2, 3)$. Then the commutator

$$(y')^{-1} s_1 y' s_1^{-1}$$

$$= \left( \cdots, x_{-1}, x_0, x_1, x_2, x_3, x_4, x_5, \cdots \right) = (2, 3, 4)$$

belongs to $M$.

**Theorem 2.** If $M = N \cup P$, where $N$ is as in Lemma 1 and $P$ is a normal subgroup of $S$, then $M$ is normal in $\Sigma$.

**Proof.** Let $y = v s$ be any element of $M$. Let $y_s = v s_1$ be any element of $\Sigma$. Then $y_s y y_s^{-1} = y_v v s_1 s_1^{-1} v_1^{-1} = v_2 v_1 s_2 v_1^{-1}$ where $v_2 \in N$, $s_2 \in P$ since $N$ is normal in $\Sigma$ and $P$ is normal in $S$. It is sufficient to show $v_1 s_2 v_1^{-1}$ is in $M$. A computation shows

$$v_1 s_2 v_1^{-1} = \left\{ h_1, \cdots, h_e, \cdots \right\} \left( x_1, \cdots, x_e, \cdots \right) \left\{ h_1^{-1}, \cdots, h_e^{-1}, \cdots \right\}$$

$$= \left\{ h_1 h_1^{-1}, \cdots, h_e h_e^{-1}, \cdots \right\} = s_2 = v_2 s_2$$

where all but a finite number of the factors of $v_3$ are $e$. Since $H/G_1$ is Abelian the product of the factors of $v_3$ is in $G_1$. Therefore, $v_3 \in N$, $s_2 \in P$ and $M$ is normal in $\Sigma$.

**Theorem 3.** If $V \supset M$, $M$ normal in $\Sigma$, $M \cap S = P = S(B, D)$, $d \leq D \leq C$, $M \cap V = N$ then $M = N \cup P$.

**Proof.** Assume there exists $y \in M$, $y = v s$, and $s \in P$. This means $s$ moves $D$ or more of the elements of $U$. Then $y$ is conjugate to $y'$ of $M$ where $y'$ in its cyclic normal form has $d$-cycles written in the form

$$\left( \cdots, x_{-1}, x_0, x_1, \cdots \right).$$

$$\left( \cdots, x_{-1}, x_0, g x_1, x_2, \cdots \right).$$

Construct $s_1$ as follows. For each $n$-cycle, $n \geq 3$, of $y'$ of the form
let $s_1$ have the cycle $(x_1, x_2)$. For each pair of 2-cycles

\[
\begin{pmatrix}
 x_1, & x_2 \\
 x_2, & bx_1
\end{pmatrix}, \quad \begin{pmatrix}
 x_3, & x_4 \\
 x_4, & cx_3
\end{pmatrix}
\]

of $\gamma'$ let $s_1$ have the cycle $(x_1, x_3)(x_2)(x_4)$. If there is a 2-cycle

\[
\begin{pmatrix}
 x_a, & x_b \\
 x_b, & dx_a
\end{pmatrix}
\]

of $\gamma'$ left over let $s_1$ send $s_a, s_b$ into themselves. For each $d$-cycle of $\gamma'$

\[
\begin{pmatrix}
 \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\
 \cdots, & x_0, & g x_1, & x_2, & \cdots
\end{pmatrix}
\]

let $s_1$ have the cycles

\[
\begin{pmatrix}
x_0 & x_n, & x_{-n} \\
x_0 & x_{-n}, & x_n
\end{pmatrix}
\]

for $n = 1, 2, \ldots$.

Form the commutator $(\gamma')^{-1}s_1\gamma'(s_1)^{-1} = \gamma_1$ which is in $M$. For each $n$-cycle, $n \geq 3$, of $\gamma'$, $\gamma_1$ contains $(x_1, x_2, x_3)$. For each pair of 2-cycles $\gamma_1$ contains $(x_1, x_3)(x_2, x_4)$. For each $d$-cycle of $\gamma'$, $\gamma_1$ will contain

\[
\begin{pmatrix}
 \cdots, & x_{-3}, & x_{-2}, & x_{-1}, & x_0, & x_1, & x_2, & x_3, & \cdots \\
 \cdots, & x_{-5}, & x_{-4}, & x_{-3}, & x_{-2}, & x_{-1}, & g^{-1}g x_{-1}, & x_0, & x_1, & \cdots
\end{pmatrix}
= \begin{pmatrix}
 \cdots, & x_2, & x_0, & x_{-2}, & \cdots \\
 \cdots, & x_0, & x_{-2}, & x_{-4}, & \cdots
\end{pmatrix}\begin{pmatrix}
 \cdots, & x_3, & x_1, & x_{-1}, & \cdots \\
 \cdots, & x_1, & x_{-1}, & x_{-3}, & \cdots
\end{pmatrix}.
\]

This shows that $M$ contains a permutation which moves the same number of $x$'s as the $s$ of $\gamma = vs$, contradicting $P = S(B, D)$.

If $\gamma = vs \in M$ then since $s \in M$, $s^{-1}$ also belongs to $M$, and $\gamma s^{-1} = v \in M$.

**Theorem 4.** Let $M \subseteq \Sigma$, $M$ normal in $\Sigma$, $M \cap S = A(B, d)$, $M \cap V = N$ and $M/N \cong A(B, d)$. Then $M = N \cup A(B, d)$.

**Proof.** Since $M \subseteq (N \cup A)$ and $N \cup A$ is normal in $\Sigma$ it follows that $N \cup A$ is normal in $M$. Thus $N \cup A/N$ is normal in $M/N$ which is simple since $M/N \cong A(B, d)$. Therefore, $N \cup A = M$.

**Theorem 5.** Let $M \subseteq \Sigma$, $M$ normal in $\Sigma$, $V \supseteq M$, $M \cap S = A$, $M \cap V = N$, and $M/N$ not $\cong A$. Then $M = N \cup A \cup L$, where $L$ is the cyclic subgroup generated by
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\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

with \( a^2 \) in \( G_1 \). If \( L_1 \) is the cyclic subgroup generated by

\[ y_1 = \begin{pmatrix} x_1, & x_2 \\ x_2, & bx_1 \end{pmatrix}, \]

where \( b^2 \in G_1 \), \( M_1 = N \cup A \cup L_1 \) is \( M \) if, and only if, \( a \) and \( b \) are in the same coset of \( G_1 \).

**Proof.** \( M \) contains an element \( y = vs \) with \( s \) in \( S(B, C) \), \( C \supseteq d \), and \( s \in A \). Otherwise, for every \( y = vs \) of \( M \), \( s \in A \). Then every element of \( V \cup M \) would be of the form \( y = v_1 vs = v_2 s \) and \( (V \cup M)/V \cong A \). But \( (V \cup M)/V \cong M/N \) not \( \cong A \). Now if \( y = vs \) with \( s \in S(B, C) \) with \( C > d \) the method used to prove Theorem 3 will lead to \( M \cap S \neq A \), a contradiction. So assume \( s \in S(B, d) \). The product of any 2 elements of \( M \) outside \( N \cup A \) is in \( N \cup A \) since the permutation component is finite and even. Let \( x_1, x_2 \) be two elements \( s \) leaves fixed. The permutations \( s^{-1}(x_1, x_2) \) belong to \( A = P \subset M \). Therefore, \( ys^{-1}(x_1, x_2) = vss^{-1}(x_1, x_2) \) belongs to \( M \). There is an element \( v_1 \) in \( N \) such that \( v_1 v(x_1, x_2) \) can be reduced to the form

\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

because the factors of elements of \( N \) are unrestricted except that the product is in \( G_1 \). This element squared is in \( N \subset M \) so \( a^2 \in G_1 \).

**Theorem 6.** Let \( M = N \cup A \cup L \), where \( N \) is as in Lemma 1 and \( L \) is the cyclic group generated by

\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

with \( a^2 \in G_1 \). Then \( M \) is normal in \( \Sigma \).

**Proof.** It is sufficient to show \( vsv(yv)^{-1} \) belongs to \( M \) for all \( vs \) of \( \Sigma \) because \( y^2 \) belongs to \( N \). This may be reduced to showing \( sys^{-1} \) and \( vsv^{-1} \) belong to \( M \). Let \( s \) be arbitrary in \( S \) and

\[ s = \begin{pmatrix} x_i, & \cdots, & x_i, & \cdots \\ x_1, & \cdots, & x_2, & \cdots \end{pmatrix}. \]

Then

\[ sys^{-1} = \begin{pmatrix} x_i, & x_i \\ x_i, & ax_i \end{pmatrix}. \]
But there exists an $s_1$ of $A$ such that
\[ s_1 = \begin{pmatrix} x_1, \cdots, x_j, \cdots \\ x_1, \cdots, x_2, \cdots \end{pmatrix}. \]
But $sy^{-1} = s_1y^{-1}$ belongs to $(A \cup L \cup A) \subseteq M$. Let $v$ be arbitrary in $V$ and $v = \{h_1, \cdots, h_t, \cdots \}$. The commutator $y^{-1}vy^{-1} = \{a^{-1}h_2ah_1^{-1}, h_1h_2^{-1}, e, \cdots \}$ belongs to $N \subseteq M$ if $a^{-1}h_2ah_1^{-1}h_1h_2^{-1}$, belongs to $G_1$. Since $H/G_1$ is Abelian the desired result follows.

**Theorem 7.** If $C = S_n$, where $n$ is finite, and if $H$ is finite then the number of normal subgroups of $\Sigma$ is finite.

**Proof.** This follows from the fact that there are only a finite number of choices of (1) normal subgroups of $S(B, C)$, (2) normal subgroups of $H$, (3) cosets of $G_1$.

**Bibliography**


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