CONTINUITY PROPERTIES OF DERIVATIVES OF SEQUENCES OF FUNCTIONS¹

G. R. MACLANE

In a recent paper Dvoretzky [1] discusses an interesting generalization of a theorem of Walsh [2]. A striking supplement to Dvoretzky's theorem is the following one.

**Theorem.** There exists a sequence of functions

\[ \{f_n\}_{n=1}^{\infty}, \quad f_n \in C^1(-\infty, \infty), \]

with \(\lim_{n \to \infty} f_n(x) = 0\), such that: if \(N_1\) is any subsequence of the natural numbers with the property that there exists a sequence \(x_{n_1}, n_1 \in N_1\), satisfying

\[(1) \quad f'_{n_1}(x_{n_1}) = 0, \quad \text{and} \quad \lim_{n_1 \to \infty} x_{n_1} = 0,\]

then the sequence \(N_2\) complementary to \(N_1\) (i.e., \(N_2\) contains exactly those natural numbers omitted by \(N_1\)) is infinite and

\[(2) \quad \limsup_{n_2 \to \infty} \int_0^h |f'_{n_2}(x)| \, dx = \infty\]

for every \(h > 0\).

**Proof.** Let \(\epsilon_n \downarrow 0\) and let \(\{\lambda_n\}\) be the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, \cdots. Let \(\{\mu_n\}\) be a sequence of positive numbers such that

\[(3) \quad \epsilon_n \mu_n / \lambda_n \to \infty, \quad n \to \infty.\]

The functions \(f_n(x)\) shall be odd and

Received by the editors September 4, 1956.

¹ This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contract No. AF 18(600)-1135.
\[ f_n(x) = \begin{cases} 
\epsilon_n \sin \lambda_n x, & |x| \leq \pi/2\lambda_n, \\
\epsilon_n \cos \mu_n \left( x - \frac{\pi}{2\lambda_n} \right), & x > \pi/2\lambda_n.
\end{cases} \]

It is easily verified that \( f_n \in C^1 \) and \( f_n(x) \to 0 \). The smallest zeros of \( f'_n(x) \) are \( \pm \pi/(2\lambda_n) \). Because of (1) and the nature of \( \lambda_n \), any possible \( N_2 \) will contain infinitely many \( n_2 \) associated with each possible value of \( \lambda \). Thus \( N_2 \) splits into disjoint infinite sequences \( M_p, p \geq 1 \), such that
\[
\lambda_{m_p} = \frac{\pi}{p}, \quad m_p \in M_p.
\]

For a given \( h > 0 \), choose \( p \) such that \( \pi/p \leq h \). Then
\[
\int_0^h |f'_{m_p}(x)| \, dx > \int_{\pi/(2p)}^{\pi/p} |f'_{m_p}(x)| \, dx
\]
\[
= \epsilon_{m_p} \mu_{m_p} \int_{\pi/(2p)}^{\pi/p} \left| \sin \mu_{m_p} \left( x - \frac{\pi}{2p} \right) \right| \, dx
\]
\[
= \epsilon_{m_p} \int_0^{\pi/2p} |\sin t| \, dt \geq \epsilon_{m_p} \left[ \mu_m / p \right],
\]
and (2) follows from this inequality and (3).

Finally we note that Dvoretzky's use of the word "clearly" in the third line after equation (6) is dubious.

**References**


*The Rice Institute*