Differential Equations Involving a Parametric Function

R. H. Cameron

1. Introduction. It is the purpose of this paper to find conditions on the function \( f(t, u) \) under which the differential system

\[
\begin{align*}
\frac{dz}{dt} + f[t, y(t) + z(t)] &= 0, \quad t \in I, \\
z(0) &= 0
\end{align*}
\]

has a solution \( z(t) \) on the unit interval \( I \) for almost all choices of the function \( y \) in the space \( C \). Here \( C \) denotes the space of functions which are continuous on the interval \( I: 0 \leq t \leq 1 \) and which vanish at \( t = 0 \); and "almost all" means all except a set of Wiener measure zero. Under the transformation

\[
z(t) = x(t) - y(t)
\]

the system (1.1) goes into the equivalent nonlinear integral equation

\[
y(t) = x(t) + \int_0^t f[s, x(s)] ds, \quad t \in I,
\]

so that we are seeking conditions on \( f \) which make (1.3) have a solution \( x \in C \) for almost every choice of \( y \) in \( C \).

The simplest conditions of this type which we have found, and which do not force (1.3) to have a solution for every \( y \in C \), are given in the following theorem.

**Theorem 1.** Let \( f(t, u) \) have continuous first partial derivations \( f_t \) and \( f_u \) in the strip \( R: 0 \leq t \leq 1, -\infty < u < \infty \), and let \( f \) satisfy the three order of growth conditions

\[
\begin{align*}
(1.4) & \quad f(t, u) \sgn u \leq -A_1 \exp(Bu^2) & \text{in } R, \\
(1.5) & \quad f_u(t, u) + 4g_t(t, u) \leq 2\alpha^2 u^2 + A_2 & \text{in } R,
\end{align*}
\]

Presented to the Society April 20, 1957; received by the editors February 15, 1957.

1 This research was supported by the United States Air Force, through the office of Scientific Research of the Air Research and Development Command, under contract No. AF 18(603)-30. Reproduction in whole or in part is permitted for any purpose of the United States Government.

2 The author wishes to thank Mr. James Yeh for carefully checking the manuscript of this paper.

3 See, for instance, [4].
(1.6) \[ g(1, u) \geq -\frac{1}{2} \alpha u^2 \cot \beta - A_3 \quad \text{for all real } u, \]

where

(1.7) \[ g(t, u) = \int_0^u f(t, v) dv \quad \text{in } R, \]

and \( A_1, A_2, A_3, B, \alpha, \beta \) are positive constants with \( \alpha < \beta < \pi \) and \( B < 1 \). Then it follows that corresponding to almost every choice of \( y \in C \), the system (1.1) has a solution \( z \in C \), (and of course, this is the only solution defined on the interval \( I \)).

The fact that this can apply in cases where classical theorems (i.e., theorems giving conditions under which there are solutions for all choices of \( y \) in \( C \)) do not apply is shown by the counter-example

\[
f(t, u) = \frac{1}{2} u^{1/3} \sin (u^{4/3}) + \frac{1}{2} u^{5/3} \cos (u^{4/3})
\]

(1.8)

\[
= -\frac{3}{8} \frac{d}{du} [u^{4/3} \sin u^{4/3}] \quad \text{in } R.
\]

It is easy to see that (1.8) satisfies the conditions of the theorem, and hence that there are solutions of (1.1) for almost all \( y \) in \( C \) when \( f \) is given by (1.8). On the other hand, we shall show in §3 that there exists at least one \( y \) in \( C \) for which (1.1) has no solution. Thus Theorem 1 cannot be contained in any classical theorem.

2. **Proof of Theorem 1.** Assume that the hypotheses of the theorem are satisfied, choose \( \gamma = B^{-1} - 1 \), and let

(2.0) \[ \phi(t, u) = (t + \gamma)^{-1/2} \exp \{ u^2(t + \gamma)^{-1} \} \quad \text{in } R. \]

Let

(2.1) \[ G(t, u, \lambda) = g(t, u) + \lambda \phi(t, u), \quad (t, u) \in R, \lambda \geq 0, \]

so that

(2.2) \[ G_u(t, u, \lambda) = f(t, u) + 2\lambda(t + \gamma)^{-1} u \phi(t, u), \quad (t, u) \in R, \lambda \geq 0 \]

and

(2.3) \[ G_{uu} + 4G_t = f_u + 4g_t, \quad (t, u) \in R, \lambda \geq 0. \]

From (2.1), (2.0), (1.4) we have

(2.4) \[ G_u \operatorname{sgn} u \geq A_1 \]
when $|u| \geq \lambda^{-1}B^{-3/2}A_1$ and $t \in I$ and $\lambda > 0$, and since for fixed positive $\lambda$, $G_u$ is bounded for $|u| \leq \lambda^{-1}B^{-3/2}A_1$, $t \in I$, it follows that there is a positive function of $\lambda$ alone, $A(\lambda)$, such that

$$G_u(t, u, \lambda) \text{ sgn } u \geq - A(\lambda), \quad (t, u) \in R, \lambda > 0.$$

Now (2.5) implies that for fixed positive $\lambda$, the integral equation

$$y(t) = x(t) + \int_0^t G_u(s, x(s), \lambda)ds$$

has a solution $x \in C$ for each $y \in C$. For the equation clearly has a solution on some interval to the right of zero, and if this interval does not include $t=1$, it must be open at the right hand end and the solution must become unbounded in the neighborhood of this point. But by (2.5), this would imply that $y$ would vary by unbounded amounts as $x$ did so, contrary to the assumption that $y$ is continuous.

Since for fixed positive $\lambda$, (2.6) has a solution for each $y$ in $C$, it follows from Theorem 3 of [3], (using Footnote 9), that

$$\int_C \exp \{J(x, \lambda)\} d\mu x = 1, \quad \lambda > 0,$$

where

$$J(x, \lambda) = \int_0^1 K(s, x(s), \lambda)ds - 2G(1, x(1), \lambda) \quad \text{for } x \in C, \lambda \geq 0,$$

and

$$K(t, u, \lambda) = \frac{1}{2} G_{u,u} - G_u^2 + 2G_t \quad (t, u) \in R, \lambda \geq 0.$$

But by (2.9), (2.2), (2.3),

$$\lim_{\lambda \to 0^+} K(t, u, \lambda) = K(t, u, 0) = \frac{1}{2} f_u + 2g_t - f^2 \quad \text{in } R,$$

and for each fixed $x$ in $C$, we have from (2.8), (2.10), (2.1), (2.2), (2.3), by bounded convergence

$$\lim_{\lambda \to 0^+} J(x, \lambda) = J(x, 0)$$

$$= \int_0^1 K(s, x(s), 0)ds - 2g(1, x(1)).$$
Moreover, it follows from (2.8), (2.9), (2.3), (2.1), (1.5), (1.6), that
\[ J(x, \lambda) \leq \frac{1}{2} \int_0^1 \{ f_u(s, x(s)) + 4g_t(s, x(s)) \} ds - 2g(1, x(1)) \]
(2.12)
\[ \leq \log Q(x) + \frac{1}{2} A_2 + 2A_3 \quad \text{for } x \in C, \lambda > 0 \]
where for \( x \in C \),
\[ Q(x) = \exp \left\{ \alpha^2 \int_0^1 [x(s)]^2 ds + \alpha[x(1)]^2 \cot \beta \right\}. \]
(2.13)
Finally, we show that \( Q(x) \) is integrable over \( C \), by applying the transformation
\[ y(t) = x(t) - \alpha \int_0^t \cot [\alpha s + \beta - \alpha] x(s) ds, \quad t \in I \]
to the Wiener integral of unity, using Theorem A of [2]. We obtain (using (2.13))
\[ 1 = \int_C 1dwx = \exp \left\{ -\frac{1}{2} \alpha \int_0^1 \cot [\alpha s + \beta - \alpha] ds \right\} \int_C Q(x) dwx, \]
so that the integrability of \( Q \) is established.

Now we take limits in (2.7) as \( \lambda \to 0^+ \), using (2.11), (2.12) and dominated convergence, and thus establish that (2.7) holds even when \( \lambda = 0 \). Hence it follows from Theorem 3 of [3], (using Footnote 9) and from (2.8), (2.9), (2.1), (2.2), (2.3), that the integral equation (1.3) has solutions \( x \in C \) for almost all \( y \in C \). But (1.3) is equivalent to (1.1) by the transformation (1.2), and the theorem is proved.

3. A counterexample. We shall now show that when \( f(t, u) \) is given by (1.8), there exists a function \( y \in C \) such that (1.3) (and hence also (1.1)), has no solution in \( C \). We begin by constructing a certain function \( x \) which does not belong to \( C \) because it becomes infinite as we approach \( t = 1 \).

Let
\[ t_n = 1 - n^{-1/3} \quad \text{and} \quad u_n = (2n\pi)^{3/4}, \quad n = 1, 2, 3, \ldots, \]
let \( M \) be the set of monotonically increasing functions defined on \([0, 1)\) which satisfy
\[ x(0) = 0, \quad x(t_n) = u_n, \quad n = 1, 2, \ldots, \]
and let \( M_c \) be the subset of \( M \) consisting of those elements of \( M \) which are continuous on \([0, 1)\). Define the functionals \( Q_n(x) \) by
\[ Q_n(x) = \int_{t_n}^{t_{n+1}} f[x(s)] ds, \quad x \in M, \]

where \( f(u) = f(t, u) \) is given by (1.8). We shall show the existence of an element \( x \) of \( M_c \) for which

\[ Q_n(x) = u_n - u_{n+1} \]

for sufficiently large \( n \); i.e., for which

\[ x(t_{n+1}) - x(t_n) + Q_n(x) = 0 \]

for sufficiently large \( n \).

To show that there is an element of \( M_c \) for which (3.4) holds for all sufficiently large \( n \), consider a particular interval \([t_n, t_{n+1}]\) and an element \( x_1 \) of \( M \) which is constant on \([t_n, t_{n+1}]\), so that

\[ x_1(t) = u_n, \quad t_n \leq t < t_{n+1}. \]

Then we have by (3.3), (3.1), (1.8),

\[ Q_n(x_1) = f(u_n)[t_{n+1} - t_n] = \frac{1}{2} u_n^\frac{5}{3}(t_{n+1} - t_n) > 0 > u_n - u_{n+1}. \]

Now it is clear that \( M_c \) is dense in \( M \) in the \( L_1[t_n, t_{n+1}] \) topology, and also that \( Q_n(x) \) is continuous in the \( L_1[t_n, t_{n+1}] \) topology applied to the space \( M \), since \( u_n \leq x(t) \leq u_{n+1} \) when \( x \in M \) and \( t \in [t_n, t_{n+1}] \). Hence there is an element \( x_2 \in M_c \) for which \( Q_n(x_2) \) differs as little as we please from \( Q_n(x_1) \), and in particular

\[ Q_n(x_2) > u_n - u_{n+1}. \]

To obtain an \( x \) where the inequality goes the other way, we now set

\[ x_3 = u_n' = [2(n + 1)\pi]^{3/4}, \quad t_n < t < t_{n+1}, \]

so that we have by (3.3), (3.1), (1.8),

\[ Q_n(x_3) = f(u_n')[t_{n+1} - t_n] \]

\[ = -\frac{1}{2} [(2n + 1)\pi]^{5/4}[n^{-1/3} - (n + 1)^{-1/3}], \]

\[ \leq -\frac{1}{6} [2n\pi]^{5/4}[n + 1]^{-4/3}, \]

while

\[ u_n - u_{n+1} = (2n\pi)^{3/4} - [2(n + 1)\pi]^{3/4} \geq -\frac{3}{4} (2\pi)^{3/4} n^{-1/4}. \]
It is clear that there exists a positive integer \( N \) such that for \( n > N \), the last member of (3.8) is greater than the last member of (3.7), so that we have

\[
Q_n(x_3) \leq u_n - u_{n+1}
\]

if \( n > N \).

Hence it follows from the continuity of \( Q_n \) in the \( L_1[t_n, t_{n+1}] \) topology and the density of \( M_c \) in \( M \) that there exists an element \( x_4 \in M_c \) for which

(3.9) \[
Q_n(x_4) \leq u_n - u_{n+1}
\]

if \( n > N \).

Now if we put

\[
x(t) = \lambda x_2(t) + (1 - \lambda)x_4(t), \quad t \in I,
\]

it follows from (3.6) and (3.9) and the continuity of \( Q_n(x) \) that there is a value of \( \lambda \) on \( (0, 1) \) for which (3.4) holds, if \( n > N \). Thus for \( n > N \), it is possible to choose \( x(t) \) on each interval \( [t_n, t_{n+1}] \) so as to make (3.4) hold, and these choices can be made independently for each \( n > N \). Let such choices be made for each interval \( [t_n, t_{n+1}] \), \( n > N \), and choose \( x \) on the previous intervals in any way which makes \( x \in M_c \).

Using this choice of \( x \), we now define \( y \) on \([0, 1)\) by substituting this \( x \) in (1.3). It now follows from (3.4) that for \( n > N \), (3.5) holds, and from (3.3), (3.5), (1.3) that \( y(t_n) \) is independent of \( n \) for \( n > N \). Setting \( y(1) \) equal to this constant value, we have

(3.10) \[
y(t_n) = y(1), \quad n > N,
\]

and \( y(t) \) is now defined everywhere on \( I \).

To show that \( y \) is left continuous at \( t = 1 \), we assume \( n > N \) and \( t_n \leq t \leq t_{n+1} \), and write, using (3.10), (1.8), (3.1) and the monotonicity of \( x \),

\[
| y(t) - y(1) | = | y(t) - y(t_n) |
\]

\[
\leq x(t_{n+1}) - x(t_n) + \int_{t_n}^{t_{n+1}} | f(x(s)) | \, ds
\]

\[
\leq u_{n+1} - u_n + \int_{t_n}^{t_{n+1}} \frac{5}{3} u_{n+1} \, ds
\]

\[
= [(2n + 1)\pi]^{3/4} - (2n\pi)^{3/4} + [2(n + 1)\pi]^{5/4} [n^{-1/3} - (n + 1)^{-1/3}]
\]

\[
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

and it follows that \( y \in C \). Moreover, this choice of \( y \) corresponds to no solution \( x \in C \) of (1.3), since the solution is the chosen \( x \) and is unique on each interval \([0, 1 - \epsilon]\), and any solution on \([0, 1]\) would have to
agree on $[0, 1)$ with this unbounded $x$ that we chose. Thus it is established that if $f$ is given by (1.8), the equation (1.3) does not have a solution for every $y$ in $C$.

4. Conclusion. For simplicity, we did not state Theorem 1 in its most general form, and it is easy to see from the proof of the theorem that we can weaken two of the hypotheses a little.

**Theorem 2.** If we weaken conditions (1.4) and (1.5) of Theorem 1 by replacing them by conditions

\[(1.4') f(t, u) \text{ sgn } u \geq -A_1 \exp \left( \frac{u^2}{t + \gamma} \right) \text{ in } R,\]

\[(1.5') f_u(t, u) + 4g(t, u) \leq 2\alpha^2 u^2 + A_2 + \left\{ \max [0, f(t, u) \text{ sgn } u] \right\}^2 \text{ in } R,\]

where $\gamma$ is a positive constant, it follows that the conclusion of Theorem 1 holds.

Whether or not Theorem 2 is really more general than Theorem 1 is still an open question, as the author has not yet found any function which satisfies the hypotheses of Theorem 2 but not those of Theorem 1.

In a previous paper, [1], the author raised the question whether

\[y(t) = x(t) + \int_0^t [x(s)]^2 ds\]

has solutions for almost every $y$ in $C$, and he pointed out two other questions which are equivalent to this one. These questions are not answered by this paper, since $f(t, u) = u^2$ does not satisfy condition (1.6). They have, however, recently been answered in the negative in an unpublished paper by D. A. Woodward.

**References**


University of Minnesota