(23) \[ m(a, x) = (V'U'g_a, g_b) \]

from the extension of (18). Since \( b \) is arbitrary, it follows that \( m(a, x) = V'U'g_a \) almost everywhere. In a similar manner, starting from (8), we can also prove that \( n(a, x) = V'Ug_a \) almost everywhere.

References


McGill University

---

**A PROPERTY OF THE LAPLACE TRANSFORMATION**

P. G. ROONEY

1. Introduction. While certain of the properties of the Laplace transformation are so well known that they have become engineering tools, there are others that have received very little attention, and yet are very interesting. One of these comes about as follows. Let \( f(s) \) be the Laplace transform of \( \phi(t) \), that is

\[
\mathcal{L}(\phi(t); s) = \int_0^\infty e^{-st} \phi(t) dt = \mathcal{L}(\phi(t); s).
\]

Then under certain conditions,

\[
\mathcal{L}(\phi(t^2); s) = \frac{s}{4\pi^{1/2}} \int_0^\infty e^{-\frac{y}{2}} y^{-3/2} f \left( \frac{s^2}{4y} \right) dy;
\]

this formula is given, for example, in [2, 4.1(22)]. At least one generalization of this formula is known, that giving \( \mathcal{L}(t^r \phi(t^2); s) \)—see [2, 4.1(22) and (23)]—but we propose to generalize here in a different direction, namely that of replacing the \( y^{-3/2} \) in the right-hand integral of II by \( y^{-r-1} \). Specifically, we propose to show that, under certain conditions

Received by the editors November 29, 1956 and, in revised form, February 20, 1957.
Theorem. If either
(a) \( \nu \geq 0 \), and \( e^{-t^{1/2}} \phi(t) \in L(0, \infty) \) for some \( \gamma > 0 \), or
(b) \( \nu < 0 \), and \( t e^{-t^{1/2}} \phi(t) \in L(0, \infty) \) for some \( \gamma > 0 \), then \( \mathcal{L}(\phi(t); s) \) exists for all \( s > 0 \), \( \int_{0}^{\infty} t^{\nu+1} K_{\nu}(st) \phi(t) \, dt \) exists for all \( s > \gamma \), and for all \( s > \gamma \), III holds.

Proof for \( \nu \geq 0 \). If \( s > 0 \) and \( t > 0 \), \( e^{-t^{1/2}} |\phi(t)| \leq e^{t^2/4s} = M_{1} \), so that
\[
\int_{0}^{\infty} e^{-t^{1/2}} |\phi(t)| \, dt \leq M_{1} \int_{0}^{\infty} e^{-t^{1/2}} |\phi(t)| \, dt < \infty, \quad \text{and} \quad \mathcal{L}(\phi(t); s) \text{ exists.}
\]

For the existence of \( \int_{0}^{\infty} t^{\nu+1} K_{\nu}(st) \phi(t^{2}) \, dt \) for each \( s > \gamma \), it suffices to show that for each \( s > \gamma \)
\[
I_{1} = \int_{0}^{\delta} t^{\nu+1} |K_{\nu}(st) \phi(t^{2})| \, dt, \quad \text{and} \quad I_{2} = \int_{R}^{\infty} t^{\nu+1} |K_{\nu}(st) \phi(t^{2})| \, dt
\]
are finite, for some \( R > \delta > 0 \).

Consider first \( I_{1} \). From \([1, \S 7.2.2(13) \text{ and (12)}]\), it follows that \( K_{\nu}(x) = O(x^{-\gamma}) \) as \( x \to 0^{+} \). Hence if \( s > 0 \) there is a constant \( M_{2} \) such that \( |K_{\nu}(x)| \leq M_{2} x^{-\gamma} \) for \( 0 \leq x \leq s \delta \). Thus
\[
\int_{0}^{\delta} t^{\nu+1} |K_{\nu}(st) \phi(t^{2})| \, dt \leq M_{2} s^{-\gamma} \int_{0}^{\delta} t \phi(t^{2}) \, dt < \infty,
\]
and hence \( I_{1} \) is finite.

Now consider \( I_{2} \). From \([1, \S 7.4.1(1)]\), \( K_{\nu}(x) \sim (\pi/2x)^{1/2} e^{-x} \) as \( x \to \infty \). Hence for each \( s > \gamma \) there is a constant \( M_{3} \) such that \( |K_{\nu}(x)| \leq M_{3} x^{-1/2} e^{-x} \) for \( x > sR \). Hence if \( s > \gamma \),
\[
\int_{R}^{\infty} t^{\nu+1} |K_{\nu}(st) \phi(t^{2})| \, dt \leq M_{3} s^{-1/2} \int_{R}^{\infty} t^{\nu+1/2} e^{-st} |\phi(t^{2})| \, dt.
\]
But since \( s > \gamma \), \( t^{\nu-1/2} e^{-(s-\gamma) t} \) is bounded for \( t \geq R \), say by \( M_{4} \). Hence
\[ t^{\gamma+1/2}e^{-\gamma t} \leq M_4 e^{-\gamma t}, \text{ and thus} \]
\[
\int_R^\infty t^{\gamma+1} |K_r(st)\phi(t^2)| \, dt \leq M_3 M_4 s^{-1/2} \int_R^\infty e^{-\gamma t} |\phi(t^2)| \, dt
\]
\[
\leq M_3 M_4 s^{-1/2} \int_0^\infty e^{-\gamma t/2} |\phi(t)| \, dt < \infty.
\]
Hence \( I_2 \) is finite and
\[
\int_0^\infty t^{\gamma+1} K_r(st)\phi(t^2) \, dt \text{ exists for } s > \gamma.
\]
But by [2, §5.16(40)],
\[
K_r(x) = 2^{\gamma-1}x^{-\gamma} \int_0^\infty e^{-\gamma y} y^{\gamma-1} \, dy.
\]
Hence
\[
\int_0^\infty K_r(st) t^{\gamma+1} \phi(t^2) \, dt = 2^{\gamma-1} s^{-\gamma} \int_0^\infty t \phi(t^2) \, dt \int_0^\infty e^{-\gamma s^2 t^2/4u} y^{\gamma-1} \, dy
\]
\[
= 2^{\gamma-1} s^{-\gamma} \int_0^\infty e^{-u} y^{-1} \, dy \int_0^\infty e^{-s^2 t^2/4u} \phi(t^2) \, dt
\]
\[
= 2^{\gamma-2} s^{-\gamma} \int_0^\infty e^{-u} y^{-1} \, dy \int_0^\infty e^{-s^2 u/4u} \phi(u) \, du
\]
\[
(\text{where } u = t^2)
\]
the interchange of integrations being valid for \( s > \gamma \) by Fubini’s theorem.

**Proof for \( \nu < 0 \) .** The existence of \( \mathcal{L}(\phi(t); s) \) follows as in the previous case. The existence of \( \int_0^\infty t^{\nu+1} K_r(st)\phi(t^2) \, dt \) follows from proof (a) since \( K_{-\gamma}(x) = K_r(x), \text{ } x > 0, \) and the proof of formula III follows then as in the previous case.

3. Applications. We give below three examples of the use of formula III, ranging upwards in complexity. Others, more complicated, can easily be found, but these three illustrate the method amply. Each of the integrals has, of course, been evaluated before.

**Example 1.** Let \( \phi(t) = t^{\mu-1}, \mu > 0. \) Then the theorem says in this case III is valid if \( \mu + \nu > 0 \) and \( s > 0. \) From [2, 4.3(1)], \( f(s) = \Gamma(\mu)/s^\mu, \) and III yields
\[
\int_0^\infty t^{\nu + r - 1} K_\nu(st)dt = 2^{\nu + r - 2s - (\nu + r)} \Gamma(\mu) \int_0^\infty e^{-y^{\nu + r - 1}}dy = 2^{\nu + r - 2s - (\nu + r)} \Gamma(\mu) \Gamma(\nu + \nu),
\]

or changing \(st\) to \(t\),

\[
\int_0^\infty t^{\nu + r - 1} K_\nu(t)dt = 2^{\nu + r - 2s - (\nu + r)} \Gamma(\mu) \Gamma(\nu + \nu).
\]

**Example 2.** \(\phi(t) = t^{\nu/2} J_\nu(at^{1/2})\), \(a > 0\). Then from the theorem, \(\text{III}\) is valid if \(\mu > -1, \mu + \nu > -1, s > 0\). From [2, 4.14(30)],

\[
\int_0^\infty K_\nu(st)J_\mu(at) t^{\nu + r + 1}dt = 2^{\nu + r} s^{-(\nu + r + 2)} a^\mu \int_0^\infty e^{-(1 + a^2/s^2)y^{\nu + r}}dy = \frac{(2a)^\nu (2s)^r \Gamma(\mu + \nu + 1)}{(a^2 + s^2)^{\nu + r + 1}}.
\]

**Example 3.** \(\phi(t) = J_\mu(at^{1/2}) J_\mu(bt^{1/2})\), \(a > 0, b > 0\). Then the theorem gives that \(\text{III}\) is valid for \(\mu > -1, \mu + \nu > -1, s > 0\). From [1, 7.7.3(25)]

\[
f(s) = s^{-1} e^{-(a^2 + b^2)/4s} J_\mu(ab/2s),\] and \(\text{III}\) yields

\[
\int_0^\infty t^{\nu + 1} K_\nu(st) J_\mu(at) J_\mu(bt)dt = 2s^{-(\nu + 2)} \int_0^\infty e^{-(a^2 + b^2 + s^2)y^{1/2}} y^{\nu} I_\mu\left(\frac{2ab}{s^2}\right)dy.
\]

This last integral can be evaluated by expanding \(I_\mu(x)\) in its power series thus giving

\[
\int_0^\infty t^{\nu + 1} K_\nu(st) J_\mu(at) J_\mu(bt)dt = \frac{(ab)^\nu (2s)^r}{(a^2 + b^2 + s^2)^{\nu + r + 1}} \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + 1)}
\cdot {}_2F_1\left(\frac{\mu + \nu}{2} + 1, \frac{\mu + \nu + 1}{2}; \nu + 1; \frac{4a^2b^2}{(a^2 + b^2 + s^2)^2}\right).
\]

**References**


**University of Toronto**