CONCERNING REAL VALUED MAPS OF THE $n$-SPHERE

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A definition for the width of a closed curve, believed to be new, is used together with some standard homology theory, to prove the following theorem: Let $f: S^n \to E^1$ be continuous, $0 \leq d \leq 2$, $p$ the covering map $S^n \to P^n$. If $X_d = \{x \in S^n \mid$ there exists $y \in S^n$ with $\rho(x, y) = d, f(x) = f(y)\}$, then $pX_d$ carries the nontrivial mod 2 Čech $n-1$ cycle of $P^n$. This generalizes Theorem 2 of [4], already considerably extended in other directions by Bourgin [2] and Yang [5].

We will use the following notation: $E^{n+1}$ = Euclidean $n+1$ space, $\rho$ is the Euclidean metric, $\omega$ is the origin in $E^{n+1}$. $S^n = \{x \in E^{n+1} \mid \rho(x, \omega) = 1\}$. $P^n$ is projective $n$ space. $p_i: A_1 \times A_2 \to A_i$, $i = 1, 2$ will denote the projection. $\mathcal{C}_p(A) = p$ dimensional Čech homology group of $A$ with coefficients the integers mod 2 (the only coefficients to be used here). $T^2 = S^1 \times S^1$, $\Delta$ = diagonal in $T^2$. In a space $X$, $\overline{A} = X - A$, $A$ = closure of $A$.

Lemma 1. Let $g: S^1 \to E^1$ be continuous, $X$ a connected subset of $T^2$ such that either $p_1X = S^1$ or $p_2X = S^1$. Then there exists $(x, y) \in X$ such that $g(x) = g(y)$.

Proof. If $p_1X = S^1$, then there exists $(x_1, y_1) \in X$ such that $g(x_1) = \max_{x \in S^1} g(x)$, and $(x_2, y_2) \in X$ such that $g(x_2) = \min_{x \in S^1} g(x)$. Then if neither $(g(x_1), g(y_1))$ nor $(g(x_2), g(y_2))$ is on the diagonal in $E^2$, then they are on opposite sides. If $p_1X \neq S^1$, then we may make the above argument on the second coordinate.

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Let $S$ be a metric space with metric $p$. Define, for a closed curve $\alpha: S^1 \to S$,

$$ \rho_\alpha: T^2 \to E^1 \text{ by } \rho_\alpha(\theta, \theta') = p(\alpha(\theta), \alpha(\theta')). $$

Let $A_d = \{ (\theta, \theta') \in T^2 \mid \rho_\alpha(\theta, \theta') \geq d \}$. Then define the width of the curve $\alpha$ by $w(\alpha) = \text{l.u.b.} \{ d \mid i_*: \mathcal{C}_1(A_d) \to \mathcal{C}_1(T^2) \text{ is not trivial} \}$ ($i_*$ is induced by the inclusion $i: A_d \to T^2$). [If $\alpha: S^1 \to E^2$ is convex, then $\omega(\alpha)$ is the usual minimum distance between parallel support lines.]

**Lemma 2.** $B = \{ d \mid i_*: \mathcal{C}_1(A_d) \to \mathcal{C}_1(T^2) \text{ is not trivial} \}$ is an interval in $[0, \infty)$ containing 0, or else $B$ contains only 0.

**Proof.** If $0 \leq d'' < d'$, then $A_{d''} \supset A_{d'}$. Hence if $d' \in B$, $\mathcal{C}_1(A_{d'}) \to \mathcal{C}_1(T^2)$ is nontrivial, so $d'' \in B$.

**Lemma 3.** If $0 < d \in B$, and $C_d = \{ (\theta, \theta') \in T^2 \mid \rho_\alpha(\theta, \theta') = d \}$, then $\mathcal{C}_1(C_d) \to \mathcal{C}_1(T^2)$ has for image the diagonal element $(1, 1)$.

**Proof.** Since $d \in B$, $\mathcal{C}_1(A_d) \to \mathcal{C}_1(T^2)$ is nontrivial. Since $d > 0$, $A_d \cap \Delta = \emptyset$. Since the nontrivial nondiagonal elements of $\mathcal{C}_1(T^2)$ have a nontrivial intersection with the diagonal element, the image of $\mathcal{C}_1(A_d)$ must be $(1, 1)$. Furthermore, since $\Delta \subset \text{Cl} (\widetilde{A}_d)$, the image of $\mathcal{C}_1(\text{Cl} (\widetilde{A}_d))$ in $\mathcal{C}_1(T^2)$ contains $(1, 1)$. Hence in the Mayer-Vietoris homology sequence

$$ \cdots \to \mathcal{C}_1(BdA_d) \overset{i}{\to} \mathcal{C}_1(A_d) + \mathcal{C}_1(\text{Cl}(\widetilde{A}_d)) \overset{j}{\to} \mathcal{C}_1(T^2) \to \cdots $$

if $\Gamma_1 \in \mathcal{C}_1(A_d)$ maps by inclusion into $(1, 1)$ and $\Delta_1 \in \mathcal{C}_1(A_d)$ maps by inclusion into $(1, 1)$, then $j(\Gamma_1 + \Delta_1) = 0$, so $\Gamma_1 + \Delta_1 = i \Gamma_1'$ for some $\Gamma_1' \in \mathcal{C}_1(BdA_d)$. Then the successive inclusions $\mathcal{C}_1(BdA_d) \to \mathcal{C}_1(C_d) \to \mathcal{C}_1(A_d) \to \mathcal{C}_1(T^2)$ take $\Gamma_1'$ into $(1, 1)$.

Let $T: S^n \to S^n$ be the antipodal map, $\gamma_1$ a nonbounding $T$-invariant cycle (see [5] for definitions) of some $T$-invariant triangulation of $S^n$. Then it is easily seen that there is a $\gamma_1' T \sim \gamma_1$ with $|\gamma_1'|$ a simple closed curve, $|\gamma_1'| \subset |\gamma_1|$.

**Lemma 4.** There exists a map $\alpha: S^1 \to |\gamma_1'|$ such that $w(\alpha) = 2$.

**Proof.** Let $x, -x$ be antipodal points of $|\gamma_1'|$, $\beta$ a 1:1 map of the arc $0 \leq \theta \leq \pi$ of $S^1$ onto one of the polygonal arcs of $|\gamma_1'|$ joining $x$ to $-x$. Extend $\beta$ to $\alpha: S^1 \to |\gamma_1'|$ by defining $\alpha(\theta) = \beta(\theta), \alpha(\theta + \pi) = T \circ \beta(\theta)$ for $0 \leq \theta \leq \pi$. $D = \{ (\theta, \theta + \pi) \in T^2 \mid \text{all } \theta \in S^1 \}$ is a "straight line" in $T^2$ parallel to the diagonal, so $\mathcal{C}_1(D) \to \mathcal{C}_1(T^2)$ is not trivial. Furthermore, $\rho_\alpha(\theta, \theta + \pi) = 2$ for all $\theta$, so $D = A_2$, $w(\alpha) = 2$. 

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Lemma 5. Let \( X_d = \{ x \in S^n \mid \text{there exists } y \in S^n \text{ with } \rho(x, y) = d, f(x) = f(y) \} \). Then if \( 0 \leq d \leq 2 \), and \( \gamma_1 T \sim 0 \), then \( \gamma_1 \cap X_d \neq \emptyset \).

Proof. Let \( \gamma'_i \) be as in Lemma 4, \( \alpha: S^1 \to \gamma'_i \) with \( w(\alpha) = 2 \). Then by Lemma 2 for \( 0 < d \leq 2 \), \( d \in B \), hence by Lemma 3, the image of \( C_d \) in \( C_1(T^2) \) is the diagonal element. \( C_d \) contains a continuum \( C'_d \) such that the image of \( C_1(C'_d) \) in \( C_1(T^2) \) is the diagonal element, and thus \( p_1 C'_d = S^1 \). Therefore, by Lemma 1, there exists \( (\theta, \theta') \in C_d \) such that \( f \circ \alpha(\theta) = f \circ \alpha(\theta') \), using \( g = f \circ \alpha \). Hence \( \alpha(\theta) \) and \( \alpha(\theta') \) are points of \( \gamma'_i \cap X_d \subset \gamma_i \cap X_d \).

Proof of Theorem. Let \( U \) be any open subcomplex of a triangulation \( K \) containing \( pX_d \) in \( P^n \). By Lemma 5, no nonbounding 1-cycle of \( K \) is carried by \( \bar{U} \). Hence, by the Pontrjagin Removing Theorem, \( [1] \) there exists a 1-cocycle \( \gamma^0 \), noncobounding on \( K \) and zero on \( \bar{U} \). Hence, the dual \( n - 1 \) cycle \( \gamma_{n-1} \) is nonbounding in \( K \), and carried by the dual subdivision of \( U \). By the continuity of the Čech theory \( [3] \), the theorem now follows.

Remarks. One may obtain as a corollary the following: Let \( f_1, \ldots, f_n \) be maps of \( P^n \to E^n \), \( d_1, \ldots, d_n \) numbers with \( 0 \leq d_i \leq 1 \). Let \( P^n \) be metrized by \( S(x, y) = \rho(p^{-1}x, p^{-1}y) \). Then there exist points \( x_0, x_1, \ldots, x_n \in P^n \) such that \( f_i(x_0) = f_i(x_i) \) and \( S(x_0, x_i) = d_i \).

Notice that a considerable amount of information about \( X_d \) was not used in obtaining our theorem. While it was proved that every curve \( \alpha \) of width \( \geq d \) must intersect \( X_d \), we used only curves of width 2 for the theorem. To get stronger results, one may have to abandon homology methods.

Bibliography