ON CO-RECURSIVE ORTHOGONAL POLYNOMIALS

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1. Introduction. Let \( P_n(x) \) be a polynomial of degree \( n \) \((n = 0, 1, \ldots)\) with leading coefficient unity. Then (Favard [2]) a sufficient (as well as necessary) condition for the \( P_n(x) \) to be real orthogonal polynomials is that they satisfy a recurrence formula of the form

\[
P_n(x) = (x + b_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 2, 3, \ldots
\]

where \( P_0(x) = 1, P_1(x) = x + b_1, b_n \) real, \( \lambda_n > 0 \).

D. Dickinson [1] has shown that a relation of the form

\[
P_n(x)P_{n-1}(-x) + P_n(-x)P_{n-1}(x) = 0, \quad n = 1, 2, \ldots,
\]

is equivalent to (1.1) with \( b_n = 0 \) \((n \geq 2)\) and \( P_1(0) \neq 0 \). The condition \( b_n = 0 \) \((n \geq 2)\) suggests the symmetric case, (i.e., \( P_n(-x) = (-1)^n P_n(x) \)) but this is denied by the condition \( P_1(0) \neq 0 \). (In fact, (1.2) shows that \( P_n(-x) \neq 0 \) whenever \( P_n(x) = 0 \).) It then seems natural to ask what relations exist between a set of polynomials satisfying (1.2) and the corresponding symmetric polynomials which would be obtained from the equivalent relation (1.1) if the condition \( P_1(0) \neq 0 \) were replaced with \( P_1(0) = 0 \).

We are thus led to consider the following more general situation. Given the \( P_n(x) \) satisfying (1.1) (without the restriction \( b_n = 0 \)), let the "co-recursive" polynomials \( P^*_n(x) = P^*_n(x, c) \) be defined by

\[
P^*_n(x) = (x + b_n)P^*_{n-1}(x) - \lambda_n P^*_{n-2}(x), \quad n = 2, 3, \ldots
\]

where \( P^*_0(x) = 1, P^*_1(x) = P_1(x) - c. \) We then seek to determine the properties of the \( P^*_n(x) \) from those of the \( P_n(x) \). In order to have orthogonal polynomials, we will restrict \( c \) to be real.

2. Some formal relations. Since \( P_n(x) \) and \( P^*_n(x) \) satisfy identical recurrence formulas, except for initial conditions, it follows that \( P^*_n(x) - P_n(x) \) also satisfies (1.1) together with the initial conditions, \( P^*_0(x) - P_0(x) = 0, P^*_1(x) - P_1(x) = -c. \) Therefore we have

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1 Sherman [3] noted a relation of this type for certain orthogonal polynomials associated with a Stieltjes continued fraction.
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\[ P_n(x) = P_n(x) - cQ_{n-1}(x), \quad n = 0, 1, \ldots \]

where \( Q_n(x) \) is a polynomial of degree \( n \), satisfying

\[ Q_n(x) = (x + b_{n+1})Q_{n-1}(x) - \lambda_{n+1}Q_{n-2}(x), \quad n = 1, 2, \ldots \]

where \( Q_0(x) = 0, Q_1(x) = 1 \).

Comparison of the recurrence formulas shows that \( Q_{n-1}(x)/P_n(x) \) and \( Q_{n-1}(x)/P_n^*(x) \) are the \( n \)th convergents, respectively, of the continued fractions

\[ K(z) = \frac{1}{z + b_1 - \frac{\lambda_2}{z + b_2 - \frac{\lambda_3}{z + b_3}} - \cdots}, \]

\[ K^*(z) = \frac{1}{z + b_1 - \frac{\lambda_2}{z + b_2 - \frac{\lambda_3}{z + b_3}}} - \cdots. \]

The \( Q_n(x) \) are the "numerator polynomials" studied by Shohat and Sherman [4] and Sherman [3]; \( Q_n(x) \) is itself the denominator of the \( n \)th convergent of the continued fraction

\[ K_1(z) = \frac{1}{z + b_2 - \frac{\lambda_3}{z + b_3 - \frac{\lambda_4}{z + b_4}} - \cdots}. \]

We will denote by \( \psi(x), \psi^*(x) \equiv \psi^*(x, c) \) and \( \psi_1(x) \), respectively, solutions of the moment problems associated with the above continued fractions and write \( F(z), F^*(z) \) and \( F_1(z) \) for the corresponding Stieltjes transforms:

\[ K(z) \sim F(z) = \int_{-\infty}^{\infty} (z - u)^{-1}d\psi(u)(z \text{ nonreal}), \text{ etc.} \]

3. Zeros. By Favard's theorem, the sets \{\( P_n(x) \), \{\( P_n^*(x) \)\} and \{\( Q_n(x) \)\} are orthogonal with respect to the distributions \( \psi(x), \psi^*(x) \) and \( \psi_1(x) \), respectively; hence the zeros of the individual polynomials are real and simple.

If we multiply (1.1) by \( P_{n-1}^*(x) \) and (1.3) by \( P_{n-1}(x) \) and subtract, we obtain

\[ P_{n-1}(x)P_n(x) - P_n^*(x)P_{n-1}(x) = \lambda_n \left[ P_{n-2}^*(x)P_{n-1}(x) - P_{n-1}^*(x)P_{n-2}(x) \right]. \]

Iterating this relation yields

\[ P_n^*(x)P_{n-1}(x) - P_{n-1}^*(x)P_n(x) = -c \prod_{k=1}^{n} \lambda_k \quad (n \geq 1, \lambda_1 = 1). \]

If \( x_{n,j} \ (j = 1, \ldots, n) \) denotes the zeros of \( P_n(x) \) in ascending order of magnitude, then \( P_n^*(x_{n,j})P_{n-1}(x_{n,j}) = -c \prod_{k=1}^{n} \lambda_k \), so \( P_n^*(x_{n,j}) \)
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alternates in sign with \( P_{n-1}(x_{n,j}) \). By the well known separation of the zeros of consecutive orthogonal polynomials, it follows that the zeros of \( P_n(x) \) and \( P_n^*(x) \) are mutually separated. Moreover, \( P_{n-1}(x_{n,n}) > 0 \), hence \( \text{sgn} \ P_n^*(x_{n,n}) = -\text{sgn} \ c \). Therefore if \( x_{n,j}^\ast (j=1, \ldots, n) \) denotes the zeros of \( P_n^*(x) \) in ascending order of magnitude, we have

\[
(3.1) \quad x_{n,j-1}^\ast < x_{n,j-1}^\ast < x_{n,j}^\ast \quad (j = 2, \ldots, n; \ c > 0),
\]

with the roles of \( x_{n,j} \) and \( x_{n,j}^\ast \) interchanged for \( c < 0 \).

Finally, let \( x_{n-1,j}^\ast \equiv y_{n,j} \ (j=1, \ldots, n-1) \) denote the zeros of \( Q_{n-1}(x) \) in ascending order. Then it is well known (e.g. [6]) that

\[
(3.2) \quad x_{n,j-1}^\ast < y_{n,j-1} < x_{n,j} \quad (j = 2, \ldots, n). \]

Combining (3.1) and (3.2), we have

**Theorem 1.** The zeros of \( P_n(x) \), \( P_n^*(x, c) \) and \( Q_{n-1}(x) \) (\( n \geq 2 \)) are mutually separated in the following manner:

\[
x_{n,j-1} < x_{n,j-1}^\ast < y_{n,j-1} < x_{n,j} \quad (j = 2, \ldots, n; \ c > 0),
\]

with the roles of \( x_{n,j} \) and \( x_{n,j}^\ast \) reversed for \( c < 0 \).

**Corollary.** The zeros of \( P_n^*(x, c) \) are increasing functions of \( c \) and the zeros of any two distinct co-recursive orthogonal polynomials of the same degree \( n (n \geq 1) \) are mutually separated.

**Proof.** Let \( P_n^*(x, c_1) \) and \( P_n^*(x, c_2) \) be any two-recursive polynomials. Let \( P_n^*(x, c_1) = P_{n,1}(x) \) and \( P_n^*(x, c_2) = P_{n,2}(x) \). Then \( P_{n,1}(x, c_1 - c_2) = P_{n,1}(x) \).

Next let \([a, b], [a^*, b^*] \) and \([a', b']\) denote the "true" intervals of orthogonality of \( \{P_n(x)\}, \{P_n^*(x)\} \) and \( \{Q_n(x)\} \), respectively; that is [4], \( a = \lim_{n \to 0} x_{n,1}, b = \lim_{n \to 0} x_{n,n} \), etc. Then by Theorem 1, for \( c > 0 \), \( a \leq a^* \leq a' < b' \leq b \leq b^* \), and for \( c < 0 \), \( a^* \leq a \leq a' < b' \leq b^* \leq b \).

Further, \( a = -\infty \) (\( b = +\infty \)) if and only if \( a' = -\infty \) (\( b' = +\infty \)) [3], hence \( a, a^*, a'(b, b^*, b') \) are all finite or all infinite. If \( a \) or \( b \) is finite, Theorem 1 shows that \( P_n^*(x) \) has at most one zero exterior to \([a, b]\), while all zeros of \( Q_n(z) \) (for all \( n \)) are interior to \([a, b]\).

**Theorem 2.** The zeros of \( P_n^*(x, c) \) are all interior to \([a, b]\) for all \( n \) if and only if

\[
(3.3) \quad \lim_{n \to \infty} \frac{P_n(a)}{Q_{n-1}(a)} = A \leq c \leq B \equiv \lim_{n \to \infty} \frac{P_n(b)}{Q_{n-1}(b)}. \]

Here \( A (B) \) must be replaced by \( -\infty (+\infty) \) in case \( a = -\infty (b = +\infty) \).
PROOF. Suppose \( a \) is finite. Then it follows from the "determinant formula" for continued fractions \([6]\) that for \( x \leq a \), \( P_n(x)/Q_{n-1}(x) < P_{n+1}(x)/Q_n(x) < 0 \) (as is known). Hence \( A \) exists and by (2.1)

\[
P_n^*(a)/Q_{n-1}(a) = P_n(a)/Q_{n-1}(a) - c < A - c \quad (n \geq 2).
\]

Thus \( P_n^*(a)/Q_{n-1}(a) < 0 \) and \( P_n^*(x) \) does not change sign for \( x \leq a \) (for all \( n \)) if and only if \( A \leq c \).

4. **On the distribution functions.** Sherman \([3]\) has shown that the complete convergence of (2.3) implies the complete convergence of (2.5). By a theorem of Carleman (cf. \([5]\)) or by direct verification, the complete convergence of (2.3) is seen to imply the complete convergence of (2.4) also. Thus if \( \psi(x) \) is a solution of a determined moment problem, and hence is uniquely determined up to an additive constant at all points of continuity \([5]\), then so is \( \psi^*(x) \) (as well as \( \psi_1(x) \)). Moreover, if we define

\[
\begin{align*}
q_n(x) &= (\lambda_1 \cdots \lambda_{n+1})^{-1/2}Q_{n-1}(x), \quad p_n(x) = (\lambda_1 \cdots \lambda_{n+1})^{-1/2}P_n(x), \\
\tilde{p}_n(x) &= (\lambda_1 \cdots \lambda_{n+1})^{-1/2}P_n^*(x),
\end{align*}
\]

then \( \sum_{n=0}^{\infty} |p_n(x)|^2 \) diverges at all points of continuity of \( \psi(x) \) and converges at a discontinuity of \( \psi(x) \)[5]. Similar interpretations hold for \( \sum_{n=0}^{\infty} |\tilde{p}_n(x)|^2 \) and \( \sum_{n=0}^{\infty} |q_n(x)|^2 \). But if (2.3) corresponds to a determined moment problem, then \( \sum_{n=0}^{\infty} |p_n(x)|^2 \) and \( \sum_{n=0}^{\infty} |q_n(x)|^2 \) cannot converge for a common value of \( x \)[7]. It follows from (2.1) that no two of the above three series can converge for a common value of \( x \), hence we have

**Theorem 3.** If \( \psi(x) \) is a solution of a determined moment problem, then so are \( \psi^*(x) \) and \( \psi_1(x) \). In this case, no two of the three distribution functions can have a common discontinuity.

The above argument shows that if, say, \( b \) is finite and \( B \neq 0 \), then \( \sum_{n=0}^{\infty} |p_n(b)|^2 \) and \( \sum_{n=0}^{\infty} |q_n(b)|^2 \) must diverge together, hence \( \psi(x) \) and \( \psi_1(x) \) must be continuous at \( b \). Similarly, \( \psi^*(x) \) is continuous at \( b \) if \( c \neq B \).

Henceforth, we assume (2.3) corresponds to a determined moment problem so that by [3], the continued fractions (2.3), (2.4) and (2.5) converge completely for all nonreal \( z \) to \( F(z), F^*(z) \) and \( F_1(z) \), respectively, with \( F(z) = [z + b_1 - \lambda_1 F_1(z)]^{-1} \) and \( F^*(z) = [z + b_1 - c - \lambda_2 F_1(z)]^{-1} \). Also,

\[
(4.1) \quad F^*(z) = F(z)/[1 - cF(z)], \quad (z \text{ nonreal}).
\]
We further assume that \( \psi(x), \psi^*(x) \) and \( \psi_1(x) \) have been normalized: \( \psi(x) = \frac{[\psi(x+0) + \psi(x-0)]}{2} \), etc. Then by the Stieltjes inversion formula

\[
(4.2) \quad \psi^*(x) - \psi^*(x_0) = \lim_{\nu \to 0^+} \frac{1}{\pi} \int_{x}^{x_0} \text{Im} \frac{F(u + iy)}{1 - cF(u + iy)} \, du.
\]

Sherman [3] has shown that if the analytic continuation of \( F(z) \) is such that \( 1/F(z) \) is regular on a segment, \([x_0, x]\), of the real axis, then

\[
(4.3) \quad \psi_1(x) - \psi_1(x_0) = -\frac{1}{\pi} \int_{x_0}^{x} \text{Im} \frac{F(u)}{1 - cf(u)} \, du.
\]

If the analytic continuation of \( F^*(z) \) is regular on \([x_0, x]\), then by the same procedure used by Sherman, we can show that

\[
(4.4) \quad \psi^*(x) - \psi^*(x_0) = \frac{1}{\pi} \int_{x}^{x_0} \text{Im} \frac{F(u)}{1 - cF(u)} \, du.
\]

If \( \psi(x) \) has an interval of constancy, \((\alpha, \beta)\), then (2.6) defines a single analytic function which is regular on the upper and lower half-planes and \((\alpha, \beta)\). If \( F(\xi) = 1/c, \alpha < \xi < \beta \), then \( F^*(z) \) has a pole at \( \xi \) and, in general, \( \psi^*(x) \) will have a jump at \( \xi \) which can be determined as follows.

We can choose \( x_0 < \xi < x \) so that \([x_0, x] \subset (\alpha, \beta)\) and \( F(u) \neq 1/c \) for \( u \neq \xi, u \in [x_0, x] \). On the closed rectangular contour \( \gamma \), whose vertices are \( x+iy, x_0+iy, x_0-iy, x-iy (y>0) \), we have by the residue theorem

\[
\int_{\gamma} [F^*(u + iy) - F^*(u - iy)] \, du + i \int_{-\gamma} [F^*(x + iv) - F^*(x_0 + iv)] \, dv = 2\pi i \text{Res} F^*(z),
\]

\[
\frac{1}{\pi} \int_{\gamma} \text{Im} F^*(u + iy) \, du = \text{Res} F^*(z) + \frac{1}{\pi} \int_{0}^{\gamma} \text{Re} [F^*(x + iv) - F^*(x_0 + iv)] \, dv.
\]

Since \( F^*(z) \) is bounded on \( \gamma \), we obtain on letting \( y \to 0^+ \),

\[
(4.5) \quad \psi^*(x) - \psi^*(x_0) = \text{Res} F^*(z).
\]

If follows that if \( \psi(x) \) is stepwise constant in an interval \((\alpha, \beta)\), then so is \( \psi^*(x) \) (and \( \psi_1(x) \) since a similar procedure clearly applies to
Moreover, if \( c \in [A, B] \), say \( B < c < \infty \), then \( b < b^* \) by Theorem 2 and \( x_{n,n-1}^* < b, \lim_{n \to \infty} x_{n,n}^* = b^* \). Since every point of increase of \( \psi^*(x) \) is a limit point of zeros of \( \{ P_n^*(x) \} \) [5], it follows that \( \psi^*(x) \) is constant for \( b < x < b^* \), hence \( \psi^*(x) \) must have a jump at \( b^* \) which can be determined by (4.5).

5. Examples. The Tchebichef polynomials furnish simple examples for which \( P_n^*(x) \) can be determined explicitly and for which \( \psi^*(x) \) is of "mixed type," that is, \( \psi^*(x) \) is absolutely continuous on part of the orthogonality interval and is a step function (with a single jump) on the remainder.

(a) \( P_n(x) = 2^{-n} U_n(x) = 2^{-n} \sin (n+1)\theta / \sin \theta, \ x = \cos \theta \). We have \( b_1 = b_n = 0, \lambda_n = 1/4 \ (n \geq 2) \); \( F(z) = 2/(z + (z^2 - 1)^{1/2}) \) (cf. [3]). Clearly, \( Q_n(x) = P_n(x) \) so by (2.1), (4.1) and (4.4),

\[
P_n^*(x, c) = 2^{-n} [U_n(x) - 2cU_{n-1}(x)]
\]

and\(^3\)

\[
\psi^*(x) - \psi^*(x_0) = \frac{2}{\pi} \int_{x_0}^x \frac{(1 - t^2)^{1/2}}{4c^2 + 1 - 4ct} \, dt, \quad -1 < x_0 < x < 1.
\]

Here \( -A = B = 1/2 \) so \( \psi^*(x) \) is constant for all \( x \in (-1, 1) \) except for \( |c| > 1/2 \) when it has a jump at \( \xi = c + 1/4c \) which is found by (4.5) to be \( 1 - 1/4c^2 \). \( \psi^*(x) \) is continuous at \( \pm 1 \) except possibly when \( c = \pm 1/2 \). However, in this case, \( P_n^*(x, c) \) reduces to a Jacobi polynomial:

\[
P_n^*(x, c) = 2^n (n!)^2 / (2n)! P_n^{(c,-c)}(x), \quad c = \pm 1/2.
\]

(b) \( P_n(x) = 2^{1-n} T_n(x) = 2^{1-n} \cos n\theta \). Here \( b_n = 0 \ (n \geq 1), \lambda_n = 1/4 \) for \( n \geq 3 \) but \( \lambda_2 = 1/2 \); \( F(z) = (z^2 - 1)^{-1/2} \) [3]. Thus \( Q_n(x) = 2^{-n} U_n(x) \) and we have

\[
P_n^*(x, c) = 2^{1-n} [T_n(x) - cU_{n-1}(x)],
\]

\[
\psi^*(x) - \psi^*(x_0) = \frac{1}{\pi} \int_{x_0}^x \frac{(1 - t^2)^{1/2}}{1 + c^2 - t} \, dt, \quad -1 < x_0 < x < 1.
\]

\( A = B = 0 \), hence for all \( c \neq 0 \), \( \psi^*(x) \) is continuous at \( \pm 1 \) and constant for \( x \in (-1, 1) \) except for a jump at \( \xi = (\text{sgn} \ c) (1 + c^2)^{1/2} \) of magnitude \( |c| (1 + c^2)^{-1/2} \).

\(^2\) Sherman has shown that this is essentially the only case when \( Q_n(x) = P_n(x) \), as is also evident from the recurrence formula.

\(^3\) The constant factor is determined in accordance with \( \int_{-\infty}^\infty d\psi^*(x) = \lambda_1 = 1 \).
The modified Lommel polynomials, $P_n(x) = [2^n(v)_n]^{-1}h_{n,v}(x)$ (cf. [1]) furnish another simple example where $P_n^*(x)$ can be explicitly given. Since $b_1 = b_n = 0$, $\lambda_n = \lambda_{n,v} = [4(n+v-1)(n+v-2)]^{-1} = \lambda_{n-1,v+1}$ $(n \geq 2)$, we have $Q_n(x) = [2^n(v+1)_n]^{-1}h_{n,v+1}(x)$, $P_n^*(x, c) = [2^n(v)_n]^{-1} \cdot [h_{n,v}(x) - 2cvh_{n-1,v+1}(x)].$ $\psi^*(x)$ is a step function whose jumps can be obtained from (4.1) and (4.5) with $F(z) = J_r(1/z)/J_{r-1}(1/z)$ where $J_r(z)$ is the Bessel function of order $v$.

Remark. It may be noted that in the preceding examples, $P_n^*(x, c)$ satisfies Dickinson’s relation (1.2) (since $b_n = 0$ $(n \geq 1)$).

References


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