

A PROPERTY OF INDECOMPOSABLE CONNECTED SETS

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Introduction. Two nonempty subsets of a topological space are said to be *separated* if neither intersects the closure of the other. A set is *connected* if it is not the union of two separated sets. A connected set I is *indecomposable* if it is not the union of two connected sets, neither of which is dense in I .

In [1], Swingle raised the following question: does there exist, in the plane, an indecomposable connected set I , such that the set $I \cup \{p\}$ fails to be indecomposable for some limit point p of I ?

The purpose of this paper is to prove that the answer is negative. It is interesting to note that the plane plays an essential role here as the embedding space: if the plane is replaced by Euclidean 3-space in Swingle's question, the answer turns out to be affirmative. The construction of such an example is rather complicated, and is not included in this paper.

NOTATION. A *component* of a set is a maximal connected subset.

If X and Y are sets, $X - Y$ denotes the set of all elements of X which are not elements of Y (whether Y is a subset of X or not).

The boundary of a set X will be denoted by ∂X , and \bar{X} will denote the closure of X .

THEOREM. *Suppose I is an indecomposable connected subset of the plane and p is a limit point of I . Then $I \cup \{p\}$ is also an indecomposable connected set.*

PROOF. We will begin with three rather basic and fairly evident lemmas. Then we will prove Lemma 4, whose purpose may at first glance be obscure to the reader. The proof of Lemma 4 depends on the fact that if there are three arcs in the plane which do not contain the point p and which have only their end-points in common, then one of these arcs is separated from p in the plane by the union of the other two. Then, assuming that the theorem is false, Steps 1 and 2 of the proof will describe the construction of some rather complicated sets. In Step 3, a set J will be defined and, using Lemma 4, it will be proved that $J \neq 0$. In Step 4, an argument analogous to the proof of Lemma 4 will prove that $J = 0$, and this contradiction implies the theorem.

Suppose X is a set in the plane. We will say that X has property λ if neither X nor $I - X$ is dense in I . We will use $\partial_I X$ to denote the boundary of X with respect to I ; that is, $\partial_I X = I \cap \bar{A} \cap \bar{B}$

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where $A = I \cap X$ and $B = I - X$. It should be noticed that, since I is connected, if $A \neq 0$ and $B \neq 0$, then $\partial_I X \neq 0$.

LEMMA 1. *If a set E is a subset of I having property λ , then E is not connected.*

PROOF. Assume E is connected. Since \bar{E} has property λ and I is indecomposable, $I - \bar{E}$ cannot be connected. So $I - \bar{E}$ is the union of two separated sets A_1 and A_2 . If $\bar{E} \cup A_1$ were the union of two separated sets B_1 and B_2 with $B_2 \subset A_1$, then the sets B_2 and $B_1 \cup A_2$ would be separated; but $I = B_1 \cup B_2 \cup A_2$, and I is connected. This contradiction shows that $\bar{E} \cup A_1$ is connected. Similarly $\bar{E} \cup A_2$ is connected. Since A_1 and A_2 are open with respect to I , neither $\bar{E} \cup A_1$ or $\bar{E} \cup A_2$ is dense in I . Since I is indecomposable this is impossible and our lemma is proved.

LEMMA 2. *If X has property λ , there are disjoint open sets $Q_i(X)$ ($i = 1, 2, 3$) whose union contains $I - X$, such that $Q_i(X)$ is open in the plane, $p \notin Q_i(X)$, and $Q_i(X)$ intersects $I - X$.*

PROOF. It is clearly sufficient to prove that $I - X$ is the union of three separated sets.

In order to prove this observe that $I - X$ has property λ . By Lemma 1, then $I - X = A_1 \cup A_2$ with A_1 and A_2 separated. One of the sets A_1 or A_2 has property λ , say A_1 . Then $A_1 = B_1 \cup B_2$ with B_1 and B_2 separated. Hence $I - X = A_2 \cup B_1 \cup B_2$ and A_2, B_1 and B_2 are separated.

LEMMA 3. *Suppose $X \subset I, H \subset X \cup \{p\}$, H is connected and X is the union of two separated sets A and B . Then $E = (A \cap H) \cup \{p\}$ is connected.*

PROOF. If $A \cap H = 0$ then the theorem is trivially correct. If $A \cap H \neq 0$, suppose E is the union of two separated sets E_1 and E_2 and $p \in E_1$. Then E_2 and B are separated. But $H = E_1 \cup E_2 \cup (B \cap H)$, and E_2 is separated from $E_1 \cup (B \cap H)$. Since H is connected, this is impossible.

LEMMA 4. (1) *Suppose U and V are disjoint connected open sets in the plane having property λ and not containing p .*

(2) *Suppose $X = (I \cup \{p\}) - (U \cup V)$ is connected.*

Then, for some $i, Q_i(U) \cap \bar{V} \cap I = 0$. ($Q_i(U)$ is defined in accordance with Lemma 2, and i is 1, 2, or 3.)

PROOF. Assume that, for each $i, I \cap Q_i(U) \cap \bar{V} \neq 0$. If $I \cap \bar{U} \cap \bar{V} \cap Q_i(U)$ exists let x_i be a point of this set. Otherwise for each i , let

T_i denote a component of $[Q_i(U) - \bar{U}]$ intersecting $\bar{V} \cap I$. Since $I \cap \partial T_i \neq \emptyset$ and $I \cap \partial T_i \subset \partial U$, we can choose, for each i , a point $x_i \in I \cap \partial T_i \cap \partial U$. Also choose $x \in U$ and $y \in V$.

Letting Q_i be the component of $Q_i(U)$ which contains x_i , it is clear that $U \cap Q_i \neq \emptyset$, and $V \cap Q_i \neq \emptyset$. One can therefore construct arcs L_i ($i=1, 2, 3$) with the following properties:

- (a) L_i has x and y as end-points.
- (b) $L_i \subset U \cup Q_i \cup V$, and $x_i \in L_i$.
- (c) If $i \neq j$, then $L_i \cap L_j = \{x\} \cup \{y\}$.

The plane is separated by $L_1 \cup L_2 \cup L_3$ in such a way that p is separated from one of these arcs by the other two. Say, p is separated from L_1 by $L_2 \cup L_3$. Then consider the set $A = Q_1(U) - (V \cup U)$. By Lemma 3 and assumption (2), $A \cup \{p\}$ is connected. But $x_1 \in A$ and $A \cup \{p\}$ does not intersect $L_2 \cup L_3$. This contradicts the connectedness of $A \cup \{p\}$, and proves the lemma.

We now turn to the proof of the theorem.

STEP 1. On the assumption that the theorem is false, we would like to show that *there are disjoint connected open sets V and W in the plane intersecting I such that*

$$H = (I \cup \{p\}) - (V \cup W) \text{ is connected.}$$

It is obvious that $I \cup \{p\}$ is connected; thus if the theorem is false, $I \cup \{p\}$ is the union of two connected sets H_1 and H_2 neither of which is dense in $I \cup \{p\}$ such that $p \in H_1$.

Since $I - \bar{H}_1$ has property λ , Lemma 1 shows that $I - \bar{H}_1$ is the union of two separated sets K_1 and K_2 . For $i=1, 2$, let G_i be open sets such that $G_i \supset K_i$, $G_i \cap H_1 = 0$, $G_1 \cap G_2 = 0$. Suppose V and W are components of G_1 and G_2 , respectively, intersecting I . In order to show that H is then connected, let us suppose that $H = A \cup B$ with A and B separated and $p \in A$. Since $H_1 \subset H$ and H_1 is connected, $H_1 \subset A$. But B is closed with respect to I and $\partial_I B \subset \bar{H}_1$. Hence A and B are not separated and this contradiction shows that H is connected.

STEP 2. We now define a well ordered sequence of plane sets F_α : Put $F_1 = W$. If α is an ordinal and F_γ has been defined for $1 \leq \gamma < \alpha$, put $S_\alpha = \bigcup_{\gamma < \alpha} F_\gamma$. If there exist separated sets A and B such that $I - S_\alpha = A \cup B$ and $A \supset \bar{V} \cap I$, then choose one such pair A, B and put $F_\alpha = B$. Let β be the first ordinal such that $I - S_\beta$ cannot be decomposed in the above manner; the well ordered sequence in question consists of the sets F_α with $1 \leq \alpha < \beta$.

For $\alpha \leq \beta$, let $G_\alpha = S_\alpha - W$; and $G = G_\beta$.

Let M and N be disjoint open sets not containing p such that $M \supset S_\beta$ and $N \supset (I - \bar{S}_\beta) \cup V$. Let L denote the component of N con-

taining V . For $\alpha < \beta$, let M_α be an open subset of M such that $M_\alpha \supset G_\alpha$ and $(I \cap \overline{M}_\alpha) - W = \overline{G}_\alpha \cap I$.

We would like to point out the following facts.

1. If $U \subset S_\beta$, $Y \subset I - U$, $Y \cap \overline{V} = 0$, and Y and $(I - U) - Y$ are separated, then $Y \subset S_\beta$. If $Y - S_\beta \neq 0$, then $Y - S_\beta$ and $(I - S_\beta) - Y$ are separated, which contradicts the definition of β .

2. For $\alpha \leq \beta$, $G_\alpha \cup \{p\}$ is connected. This can be seen by supposing γ is the smallest ordinal such that $G_\gamma \cup \{p\}$ is not connected. Clearly γ is not a limit ordinal since in this case $G_\gamma = \bigcup_{\delta < \gamma} G_\delta$. Also $\gamma \neq 1$, since $G_1 = 0$. Hence $G_{\gamma-1} \cup \{p\}$ is connected and $G_\gamma \cup \{p\} = G_{\gamma-1} \cup \{p\} \cup F_{\gamma-1}$. Therefore, if $G_\gamma \cup \{p\} = A \cup B$ with A and B separated, suppose $B \supset G_{\gamma-1} \cup \{p\}$. Then $A \subset F_{\gamma-1}$ and A and $I - S_\gamma$ are separated, by the definition of $F_{\gamma-1}$. Since $I - W \subset (I - S_\gamma) \cup A \cup B$, A and $(I - W) - A$ are separated. And since $(H - \{p\}) \subset (I - W)$, A and $(H - \{p\}) - A$ are separated. Hence, by Lemma 3, $A \cup \{p\}$ is connected. But then A and B are not separated and this contradiction shows that $G_\gamma \cup \{p\}$ is connected.

STEP 3. We will now consider $Q_i(L)$, $i = 1, 2, 3$, [L is defined in paragraph 3 of Step 2 and $Q_i(L)$ is defined in Lemma 2.]

Let $J = Q_i(L) \cap I \cap (\overline{G} - G)$. In this step we will show that $J \neq 0$.

Suppose $J = 0$. Let U denote a component of $Q_i(L) \cap M$. In order to use Lemma 4 we would like to show that $X = (I \cup \{p\}) - (U \cup V)$ is connected.

First let us show that $H \cap X$ is connected. Since $J = 0$, $U \cap G$, and $I - U$ are separated; hence $(X \cap H) - \{p\} = A$ and $B = (U \cap G)$ are separated. But then $H - \{p\} = A \cup B$, so by Lemma 3, $H \cap X = A \cup \{p\}$ is connected.

Suppose $X = D \cup E$, D and E are separated, and $H \cap X \subset D$. But $\partial_I E \subset \partial_I X \subset (I \cap (\partial U \cup \partial V)) \subset H$ since $I \cap \partial U \cap W = 0$ and $U \cap V = 0$. Therefore $\partial_I E \subset H$ and since E is closed relative to I this contradicts the separation of D and E . Hence X is connected.

We can now apply Lemma 4 and conclude that, for some j , $Q_j(U) \cap \overline{V} \cap I = 0$.

Let $Y = (Q_j(U) \cap I) - U$. Then $U \subset S_\beta$, $Y \subset I - U$, $Y \cap \overline{V} = 0$ and Y and $(I - U) - Y$ are separated. Therefore by 1 of Step 2, $Y \subset S_\beta$. Choose $y \in \partial_I Y$. There is an open disk lying in $Q_i(L) \cap M$ containing y and, since $y \in \partial U$, this disk intersects U . But since U is a component of $Q_i(L) \cap M$, the disk is in U . This contradiction shows that $J \neq 0$.

STEP 4. We will now reach a contradiction by a proof somewhat similar to that of Lemma 4.

For $i = 1, 2, 3$, let β_i denote the smallest ordinal such that

$J \cap \overline{G_{\beta_i}} \neq \emptyset$. Observe that β_i is a limit ordinal since for nonlimit ordinals α , $G_\alpha = G_{\alpha-1} \cup F_{\alpha-1}$ and $\overline{F_{\alpha-1}} \cap (\overline{G} - G) = \emptyset$. Choose $y_i \in J \cap \overline{G_{\beta_i}}$.

Suppose D_{in} is an open disk containing y_i of radius $1/n$. Then, since β_i is a limit ordinal, there is an ordinal $\alpha_{in} < \beta_i$ such that either

(a) there is a point $W_{in} \in D_{in} \cap G_{\alpha_{in}} \cap \partial W$ or

(b) there is a component $R_{in} = R$ of $(M_{\alpha_{in}} \cap Q_i(L)) - \overline{W}$ such that $R \cap G_{\alpha_{in}} \cap D_{in} \neq \emptyset$.

In case (b) $\partial_I R \cap \partial W \neq \emptyset$. This can be shown by letting γ be the smallest ordinal such that $F_\gamma \cap R \neq \emptyset$. Then

$$G_\gamma \cap R = \emptyset.$$

If $r \in \partial_I (F_\gamma \cap R)$ then $r \in \partial W$ for if $r \notin \overline{W}$, $r \in M_{\alpha_{in}} \cap Q_i(L)$ and is hence an interior point of R . Since $\partial_I F_\gamma \subset \overline{W} \cup \overline{G_\gamma}$, if $r \notin \overline{W}$, $r \in \overline{G_\gamma}$ and $r \in R$. Since R is open this contradicts the fact that $G_\gamma \cap R = \emptyset$. Hence in case (b) there is a point $w_{in} \in \partial_I R \cap \partial W$.

The following fact about the plane will be used without proof in the following two paragraphs. If W is a connected open set, w a point of W , Z a finite subset of ∂W , and ϵ is a positive number, then for each point z of Z there are arcs $L(z)$ from w to z and open disks $D(z)$ containing z such that,

- (1) if $x \neq z$, $L(z) \cap L(x) = \{w\}$,
- (2) if $x \neq z$, $D(z) \cap (L(x) \cup D(x)) = \emptyset$,
- (3) $L(z) \subset W \cup D(z)$,
- (4) the diameter of $D(z)$ is less than ϵ .

For $i = 1, 2, 3$ let w_i be one limit point of the sequence $w_{i1}, w_{i2}, w_{i3}, \dots$. [If necessary, embed the plane in a bigger plane.] Choose a point w of W and let Z be the set consisting of w_1, w_2, w_3 and those of the points y_1, y_2 , and y_3 which are in ∂W . Then for each z in Z there are arcs $L(z)$ from w to z and open disks $D(z)$ containing z such that (1), (2), and (3) of the preceding paragraph hold and (4) if $z = y_i$ for some i , then $D(z) \subset Q_i(L)$. For $y_i \notin \partial W$, let $D(y_i)$ be a disk such that the closures of W and $D(y_i)$ are disjoint and $D(y_i) \subset Q_i(L)$.

For $i = 1, 2, 3$ choose an integer m such that $w_{im} \in D(w_i)$ and $D_{im} \subset D(y_i)$; let w^i denote w_{im} , α_i denote α_{im} , and x_i denote (a) w^i if $w^i \in D(y_i)$ or (b) a point of $R_{in} \cap D(y_i)$, if $w^i \notin D(y_i)$. As before, for $i = 1, 2, 3$ there are arcs L^i from w to w^i and open disks D^i containing w^i , such that (1), (2) and (3) above follow and (3) $D^i \subset (M_{\alpha_i} \cap Q_i(L))$ and for $i \neq j$, $D^i \cap D(y_j) = \emptyset$.

There is an open set S such that $S \supset (I - S_\beta) \cup V$, $S \cap ((M_{\alpha_i} \cap Q_i(L)) \cup L_i) = \emptyset$ for all i , and $p \notin S$. Such a set clearly exists since L^i is a closed subset of $(M_{\alpha_i} \cap Q_i(L)) \cup W$; $((I - S_\beta) \cup V) \cap (M_{\alpha_i} \cup W) = \emptyset$ for all i ; and, for $x \in \partial((I - S_\beta) - V)$, $x \in Q_i(L) - W$ for some i so, for $j \neq i$, x and

$Q_j(L)$ are separated and, by the definition of β_i and the fact that $\alpha_i < \beta_i$, we have that $x \in \overline{M_{\alpha_i}}$.

Let T be the component of S containing V . Then $T \supset I - S_\beta$ for if $A = (S - T) \cap (I - S_\beta) \neq 0$ then A and $T \cap (I - S_\beta)$ are separated since they are in different components of the open set S . Since $A \cap V = 0$, this contradicts the definition of S_β . Choose $t \in T$.

Therefore, as in the proof of Lemma 4, there are arcs L_i ($i = 1, 2, 3$) having the following properties:

- (a) L_i has w and t as its end points,
- (b) $L_i \subset (W \cup (M_{\alpha_i} \cap Q_i(L)) \cup D(y_i) \cup T)$, $x_i \in L_i$,
- (c) if $i \neq j$, then $L_i \cap L_j = \{w\} \cup \{t\}$.

Using Lemma 3 and fact (2) of Step (2) we can see that $(G_{\alpha_i} \cap Q_i(L)) \cup \{p\}$ is connected since $G_{\alpha_i} \subset (Q_1(L) \cup Q_2(L) \cup Q_3(L))$ and $Q_1(L)$, $Q_2(L)$, and $Q_3(L)$ are separated.

The plane is separated by $L_1 \cup L_2 \cup L_3$ in such a way that p is separated from one of these arcs by the other two. Say p is separated from L_1 by $L_2 \cup L_3$. But since $x_1 \in L_1 \cap G_{\alpha_1} \cap Q_1(L)$ and since $(L_2 \cup L_3) \cap ((G_{\alpha_1} \cap Q_1(L)) \cup \{p\}) = 0$, this contradicts the connectedness of $(G_{\alpha_1} \cap Q_1(L) \cup \{p\})$; and proves our theorem.

REFERENCE

1. Paul M. Swingle, *The closure of types of connected sets*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 178-185.

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