A NOTE ON DOEBLIN'S CENTRAL LIMIT THEOREM

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Consider a Markov chain with a countable infinity of states (1, 2, ...) forming a single positive recurrent class, so that all the mean recurrence-times $m_{jj}$ are finite. Let $f(\cdot)$ be any real-valued function with the positive integers as domain, and if $X_m$ denotes the state of the system at the $m$th epoch, let $S_n = \sum_{m=0}^{n} f(X_m)$. Let $i$ be the initial state and $j$ any state, and let the state $j$ be occupied at the epochs $0 \leq v_1 < v_2 < v_3 < \cdots$ and at no others (this sequence will be nonterminating with probability one). We shall say that $\mu_j$ exists if $|f(X_{v_1+1}) + \cdots + f(X_{v_2})|$ has a finite expectation, and we then write

$$\mu_j \equiv \mathbb{E} \left( \sum_{m=v_1+1}^{v_2} f(X_m) \right).$$

Chung [2] has proved\(^1\) that if $\mu_j$ exists for one $j$, then it exists for all $j$. When the $\mu$'s exist he has further shown that $S_n/n$ converges in probability (as $n \to \infty$) to the limit $M \equiv \mu_j/m_{jj}$ (which is therefore independent of $j$). Now write

$$Z_{v+1} \equiv \sum_{m=v+1}^{v_2+1} f(X_m) - M(v_{v+1} - v_{v});$$

the $Z$'s will be independent random variables having a common distribution and a finite first absolute moment and we shall have $\mathbb{E}(Z) = 0$, if the $\mu$'s exist. Put $\sigma_j^2 \equiv \text{var}(Z) \leq \infty$. Chung has shown\(^2\) that $\sigma_1$, $\sigma_2$, $\sigma_3$, \ldots are either all finite or all infinite. Finally we have Doeblin's central limit theorem (Doeblin [4]; see also Chung [2]):

*If the $\mu$'s exist and the $\sigma$'s are finite and non-zero and if the recurrence times have finite second moments then $S_n$ is asymptotically normally distributed with mean $Mn$ and with variance

$$B_n \equiv \sigma_j^2 n / m_{jj}$$

(thus $B$ is independent of $j$).*

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\(^1\) In his proof one must replace the first two sentences of p. 409 by "It is easy to show that if $X$ and $Y$ are independent then $\mathbb{E}(|X+Y|) < \infty$ implies $\mathbb{E}(|X|) < \infty$. Now $W_1$, $W_2$ and $W_3$ are independent and so, by (21), $\mathbb{E}(|W_3|) < \infty."$

\(^2\) Note that his proof (second part of his Theorem 3) should be preceded by the proof of his Theorem 4, because it requires the fact that $M$ does not depend upon $j$. If this precaution is not observed, an otherwise avoidable reference has to be made to the second moment of the recurrence-time.
The purpose of this note is to show that:

In the statement and proof of Doeblin's central limit theorem the condition set out above in block capitals can be omitted.

We follow Chung by defining \( l(n) \) to be the integer \( l \) such that \( v_l \leq n < v_{l+1} \) (with \( l \equiv 0 \) if \( n < v_1 \)), and we then write, as he does,

\[
S_n - Mn = Y' + Y'' + (Z_1 + Z_2 + \cdots + Z_{l-1}) - M(n - v_l + v_1)
\]

(if \( n \geq v_1 \)), where

\[
Y' \equiv \sum_{m=0}^{v_1} f(X_m), \quad Y'' \equiv \sum_{m=1}^{n} f(X_m).
\]

Chung's arguments then show at once that \( Y'/n^{1/2}, Y''/n^{1/2} \) and \((n-v_l+v_1)/n^{1/2}\) tend to zero in probability when \( n \to \infty \). Thus (cf. Cramér [3, p. 254]) we have only to show that

\[
(Z_1 + Z_2 + \cdots + Z_{l-1})/n^{1/2}
\]

has asymptotically a normal distribution with mean zero and variance \( \sigma^2 \). Let \( 0 < \epsilon < 1 \) and let

\[
\lambda(n) = \left[ \frac{n}{m_{ij}} (1 + \epsilon) \right], \quad \mu(n) = \left[ \frac{n}{m_{ij}} \right] + 1, \quad \nu(n) = \left[ \frac{n}{m_{ij}} (1 - \epsilon) \right] + 1,
\]

so that \( 1 \leq \nu \leq \mu \leq \lambda \) if \( n > m_{ij}/\epsilon \), and then

\[
0 \leq (\lambda - \nu)/\mu \leq 2\epsilon.
\]

Now it is clear that \( \mu(n)/n \to 1/m_{ij} \) and that \((Z_1 + Z_2 + \cdots + Z_{l})/\mu^{1/2}\) is asymptotically normally distributed about zero with variance \( \sigma^2 \); thus the sharpened form of Doeblin's central limit theorem will be proved if we can show that

\[
\{(Z_1 + Z_2 + \cdots + Z_{l-1}) - (Z_1 + Z_2 + \cdots + Z_{\mu})\}/\mu^{1/2}
\]

tends to zero in probability as \( n \to \infty \).

Now (\( \omega \) designating a sample-point in the probability-space)

\[
\left\{ \omega: \frac{n}{m_{ij}} (1 - \epsilon) < l(n) - 1 < \frac{n}{m_{ij}} (1 + \epsilon) \right\}
\]

\[
\subseteq \{ \omega: \nu(n) \leq l(n) - 1 \leq \lambda(n) \},
\]

and \( \{l(n)-1\}/n \to 1/m_{ij} \) with probability one (and so also in probability) by the strong law of large numbers for identical components because

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The text is a translation of a mathematical paper from English to English.
\[ \frac{v_1 + (v_2 - v_1) + \cdots + (v_l - v_{l-1})}{l - 1} \leq \frac{n}{l - 1} < \frac{v_1 + (v_2 - v_1) + \cdots + (v_l - v_l)}{l - 1} \]

if \( v_2 \leq n \). Thus there exists an integer \( N(\varepsilon) \) such that

\[ \Pr \{ \nu(n) \leq l(n) - 1 \leq \lambda(n) \} \geq 1 - \varepsilon \text{ if } n \geq N(\varepsilon). \]

Now suppose that \( n \geq N(\varepsilon) \) and that \( n > m_{ji}/\varepsilon \) and that \( \delta > 0 \); then

\[ \Pr \{ |(Z_1 + \cdots + Z_{l-1}) - (Z_1 + \cdots + Z_n)| \geq \delta \mu^{1/2} \} \]

\[ \leq \varepsilon + \Pr \left\{ \max_{1 \leq t \leq \lambda - \nu} \left| \sum_{s=1}^t Z_{r+s} \right| \geq \delta \mu^{1/2} \right\} \]

\[ \leq \varepsilon + 4(\sigma_j/\delta)^2(\lambda - \nu)/\mu \]

\[ \leq (1 + 8\sigma_j^2/\delta^2)\varepsilon, \]

on using Kolmogorov's inequality (for this see, e.g., Halmos [5, p. 196]). The result is now established.

The delicate character of the above argument arises from the fact that we are concerned with the sum of a random number of random variables. Such questions have of course been treated in general; see, for example, Anscombe [1].

References