A NOTE ON DOEBLIN'S CENTRAL LIMIT THEOREM

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Consider a Markov chain with a countable infinity of states $(1, 2, \cdots)$ forming a single positive recurrent class, so that all the mean recurrence-times m_{jj} are finite. Let $f(\cdot)$ be any real-valued function with the positive integers as domain, and if X_m denotes the state of the system at the mth epoch, let $S_n \equiv \sum_{m=0}^{n} f(X_m)$. Let i be the initial state and j any state, and let the state j be occupied at the epochs $0 \leqq v_1 < v_2 < v_3 < \cdots$ and at no others (this sequence will be nonterminating with probability one). We shall say that μ_j exists if $|f(X_{v_1+1}) + \cdots + f(X_{v_2})|$ has a finite expectation, and we then write

$$\mu_j \equiv \mathcal{E}\left(\sum_{m=v_1+1}^{v_2} f(X_m) \right).$$

Chung [2] has proved[1] that if μ_j exists for *one* j, then it exists for *all* j. When the μ's exist he has further shown that S_n/n converges in probability (as $n \to \infty$) to the limit $M \equiv \mu_j/m_{jj}$ (which is therefore independent of j). Now write

$$Z_s \equiv \sum_{m=v_s+1}^{v_{s+1}} f(X_m) - M(v_{s+1} - v_s);$$

the Z's will be independent random variables having a common distribution and a finite first absolute moment and we shall have $\mathcal{E}(Z) = 0$, if the μ's exist. Put $\sigma_j^2 \equiv \mathrm{var}\,(Z) \leqq \infty$. Chung has shown[2] that $\sigma_1, \sigma_2, \sigma_3, \cdots$ are either *all finite* or *all infinite*. Finally we have Doeblin's central limit theorem (Doeblin [4]; see also Chung [2]):

If the μ's *exist and the* σ's *are finite and non-zero* AND IF THE RECURRENCE TIMES HAVE FINITE SECOND MOMENTS *then* S_n *is asymptotically normally distributed with mean* Mn *and with variance*

$$Bn \equiv \sigma_j^2 n/m_{jj}$$

(*thus* B *is independent of* j).

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[1] In his proof one must replace the first two sentences of p. 409 by "It is easy to show that if X and Y are independent then $\mathcal{E}(|X+Y|) < \infty$ implies $\mathcal{E}(|X|) < \infty$. Now W_1, W_2 and W_3 are independent and so, by (21), $\mathcal{E}(|W_2|) < \infty$."

[2] Note that his proof (second part of his Theorem 3) should be preceded by the proof of his Theorem 4, because it requires the fact that M does not depend upon j. If this precaution is not observed, an otherwise avoidable reference has to be made to the second moment of the recurrence-time.

The purpose of this note is to show that:

In the statement and proof of Doeblin's central limit theorem the condition set out above in block capitals can be omitted.

We follow Chung by defining $l(n)$ to be the integer l such that $v_l \leqq n < v_{l+1}$ (with $l \equiv 0$ if $n < v_1$), and we then write, as he does,

$$S_n - Mn = Y' + Y'' + (Z_1 + Z_2 + \cdots + Z_{l-1}) - M(n - v_l + v_1)$$

(if $n \geqq v_1$), where

$$Y' \equiv \sum_{m=0}^{v_1} f(X_m), \qquad Y'' \equiv \sum_{m=v_l+1}^{n} f(X_m).$$

Chung's arguments then show at once that $Y'/n^{1/2}$, $Y''/n^{1/2}$ and $(n - v_l + v_1)/n^{1/2}$ tend to zero in probability when $n \to \infty$. Thus (cf. Cramér [**3**, p. 254]) we have only to show that

$$(Z_1 + Z_2 + \cdots + Z_{l-1})/n^{1/2}$$

has asymptotically a normal distribution with mean zero and variance B. Let $0 < \epsilon < 1$ and let[3]

$$\lambda(n) \equiv \left[\frac{n}{m_{jj}}(1+\epsilon)\right], \ \mu(n) \equiv \left[\frac{n}{m_{jj}}\right] + 1, \ \nu(n) \equiv \left[\frac{n}{m_{jj}}(1-\epsilon)\right] + 1,$$

so that $1 \leqq \nu \leqq \mu \leqq \lambda$ if $n > m_{jj}/\epsilon$, and then

$$0 \leqq (\lambda - \nu)/\mu \leqq 2\epsilon.$$

Now it is clear that $\mu(n)/n \to 1/m_{jj}$ and that $(Z_1 + Z_2 + \cdots + Z_\mu)/\mu^{1/2}$ is asymptotically normally distributed about zero with variance σ_j^2; thus the sharpened form of Doeblin's central limit theorem will be proved if we can show that

$$\{(Z_1 + Z_2 + \cdots + Z_{l-1}) - (Z_1 + Z_2 + \cdots + Z_\mu)\}/\mu^{1/2}$$

tends to zero in probability as $n \to \infty$.

Now (ω designating a sample-point in the probability-space)

$$\left\{\omega: \frac{n}{m_{jj}}(1-\epsilon) < l(n) - 1 < \frac{n}{m_{jj}}(1+\epsilon)\right\}$$
$$\subseteq \{\omega: \nu(n) \leqq l(n) - 1 \leqq \lambda(n)\},$$

and $\{l(n) - 1\}/n \to 1/m_{jj}$ with probability one (*and so also in probability*) by the strong law of large numbers for identical components because

[3] Here $[x]$ denotes the integral part of x.

$$\frac{v_1 + (v_2 - v_1) + \cdots + (v_l - v_{l-1})}{l-1}$$
$$\leqq \frac{n}{l-1} < \frac{v_1 + (v_2 - v_1) + \cdots + (v_{l+1} - v_l)}{l} \frac{l}{l-1}$$

if $v_2 \leqq n$. Thus there exists an integer $N(\epsilon)$ such that

$$\text{pr}\left\{\nu(n) \leqq l(n) - 1 \leqq \lambda(n)\right\} \geqq 1 - \epsilon \text{ if } n \geqq N(\epsilon).$$

Now suppose that $n \geqq N(\epsilon)$ and that $n > m_{jj}/\epsilon$ and that $\delta > 0$; then

$$\begin{aligned} \text{pr}\left\{\left|(Z_1 + \cdots + Z_{l-1}) - (Z_1 + \cdots + Z_\mu)\right| \geqq \delta\mu^{1/2}\right\} & \\ \leqq \epsilon + \text{pr}\left\{2 \max_{1 \leqq t \leqq \lambda - \nu} \left|\sum_{s=1}^{t} Z_{\nu+s}\right| \geqq \delta\mu^{1/2}\right\} & \\ \leqq \epsilon + 4(\sigma_j/\delta)^2(\lambda - \nu)/\mu & \\ \leqq (1 + 8\sigma_j^2/\delta^2)\epsilon, & \end{aligned}$$

on using Kolmogorov's inequality (for this see, e.g., Halmos [**5**, p. 196]). The result is now established.

The delicate character of the above argument arises from the fact that we are concerned with the sum of a random number of random variables. Such questions have of course been treated in general; see, for example, Anscombe [**1**].

References

1. F. J. Anscombe, *Large-sample theory of sequential estimation*, Proc. Cambridge Philos. Soc. vol. 48 (1952) pp. 600–607.

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4. W. Doeblin, *Sur deux problèmes de M. Kolmogoroff concernant les chaines dénombrables*, Bull. Soc. Math. France vol. 52 (1938) pp. 210–220.

5. P. R. Halmos, *Measure theory*, New York, 1950.

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