DERIVATIONS IN PRIME RINGS

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We prove two theorems that are easily conjectured, namely: (1) In a prime ring of characteristics not 2, if the iterate of two derivations is a derivation, then one of them is zero; (2) If $d$ is a derivation of a prime ring such that, for all elements $a$ of the ring, $ad(a) - d(a)a$ is central, then either the ring is commutative or $d$ is zero.

**Definition.** A ring $R$ is called prime if and only if $xay = 0$ for all $a \in R$ implies $x = 0$ or $y = 0$.

From this definition it follows that no nonzero element of the centroid has nonzero kernel, so that we can divide by the prime $p$, unless $px = 0$ for all $x$ in $R$, in which case we call $R$ of characteristic $p$.

A known result that will be often used throughout this paper is given in

**Lemma 1.** Let $d$ be a derivation of a prime ring $R$ and $a$ be an element of $R$. If $ad(x) = 0$ for all $x \in R$, then either $a = 0$ or $d$ is zero.

**Proof:** In $ad(x) = 0$ for all $x \in R$, replace $x$ by $xy$. Then

$$ad(xy) - 0 = ad(x)y + axd(y) - axd(y) = 0$$

for all $x, y \in R$. If $d$ is not zero, that is, if $d(y) \neq 0$ for some $y \in R$, then, by the definition of a prime ring, $a = 0$.

The following lemma may have some independent interest.

**Lemma 2.** Let $R$ be a prime ring, and let $p, q, r$ be elements of $R$ such that $paqar = 0$ for all $a$ in $R$. Then one, at least, of $p, q, r$ is zero.

**Proof.** In $paqar = 0$, replace $a$ by $a + b$; using $paqar = pbqbr = 0$, we find $paqbr + pbqar = 0$, for all $a, b$ in $R$. If now $pa = 0$, then, for all $b$ in $R$, $pbqar = 0$, so that $p = 0$, or else $qar = 0$. But if $pa = 0$, then $pat = 0$ for all $t \in R$, so that $p = 0$ or $qat = 0$ for all $t$ in $R$; again $r = 0$, or else $qa = 0$. So $p = 0$ or $r = 0$ or $qa$ is zero whenever $pa$ is zero; replace $a$ by $aqr$; since $p(aqr) = 0$ for all $a \in R$, we see that $p = 0$ or $r = 0$ or $qar = 0$ for all $a \in R$. Similarly, $p = 0$ or $r = 0$ or $qar = 0$ for all $a \in R$. Assuming therefore that $p \neq 0$, $r \neq 0$, replace $a$ by $a + b$ in $qaqq = 0$ to find as before that $qabq + qbqar = 0$. In this equation, replace $b$ by $aqb$ to find

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Theorem 1. Let $R$ be a prime ring of characteristic not 2 and $d_1$, $d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation; then one at least of $d_1$, $d_2$ is zero.

Proof. $d_1d_2$ is a derivation, so

$$d_1d_2(ab) = d_1(d_2(a)b + ad_1d_2(b)).$$

However, $d_1$, $d_2$ are each derivations so

$$d_1d_2(ab) = d_1(d_2(ab)) = d_1(d_2(a)b + ad_2(b))$$

$$= d_1d_2(a)b + d_2(a)d_1(b) + d_1(a)d_2(b) + ad_1d_2(b).$$

But $d_1d_2(ab) = d_1d_2(a)b + ad_1d_2(b)$, so

$$d_2(a)d_1(b) + d_1(a)d_2(b) = 0$$

for all $a, b \in R$.

Replace $a$ by $ad_1(c)$ in (1). 

$$d_2(ad_1(c))d_1(b) + d_1(ad_1(c))d_2(b) = 0$$

for all $a, b, c \in R$.

$$d_2(a)d_1(c)d_1(b) + ad_2d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) + ad_1^2d_2(b) = 0.$$ 

Now $a(d_2(d_1(c))d_1(b) + d_1(d_1(c))d_2(b)) = 0$, since $d_2(d_1(c))d_1(b) + d_1(d_1(c))d_2(b) = 0$, which is merely equation (1) with $a$ replaced by $d_1(c)$. We are left, then, with

$$d_2(a)d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) = 0$$

for all $a, b, c \in R$.

But $d_1(c)d_2(b) = d_2(c)d_1(b)$ by (1) with $c$ replacing $a$. Then (2) becomes $d_2(a)d_1(c)d_1(b) - d_1(a)d_2(c)d_1(b) = 0$; factoring out $d_1(b)$ on the right, we have $(d_2(a)d_1(c) - d_1(a)d_2(c))d_1(b) = 0$ for all $b \in R$, for all $a, c \in R$. Lemma 1 is just what we need to tell us that $d_2(a)d_1(c) - d_1(a)d_2(c) = 0$ for all $a, c \in R$, unless $d_1$ is zero. But (1) with $c$ replacing $b$ tells us that instead $d_2(a)d_1(c) + d_1(a)d_2(c) = 0$ for all $a, c \in R$. Adding these last two equations, we find that $2d_2(a)d_1(c) = 0$, $d_2(a)d_1(c) = 0$, (since $R$ is not of characteristic 2), for all $a, c \in R$, or else $d_1$ is zero. Using Lemma 1 again with $d_2(a)$ replacing $a$, we find that $d_1$ is zero or else $d_2(a) = 0$ for all $a \in R$, i.e. $d_1 = 0$ or $d_2 = 0$.

In order to prove Theorem 2, we find it necessary to prove the following lemma.

Lemma 3. Let $R$ be a prime ring, and $d$ a derivation of $R$ such that 

$$ad(a) - d(a)a = 0$$

for all $a \in R$. Then $R$ is commutative, or $d$ is zero.

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Proof. \((a+b)d(a+b)-(d(a+b))(a+b)=0\) for all \(a, b \in R\); subtracting \(ad(a)-d(a)a+bd(b)-d(b)b=0\) from this, we arrive at \(ad(b)+bd(a)-d(a)b-d(b)a=0\) for all \(a, b \in R\). Write this as
\[
ad(b) - d(a)b = d(b)a - bd(a).
\]
Add to this \(ad(b)+d(a)b=d(ab)\) to find
\[
(3) \quad 2ad(b) = d(b)a - bd(a) + d(ab) \quad \text{for all } a, b \in R.
\]
In (3), replace \(b\) by \(ax\)
\[
2ad(ax) = d(ax)a - axd(a) + d(a^2x),
\]
or
\[
2ad(a)x + 2a^2d(x) = d(a)x + ad(x)a - axd(a) + 2ad(a)x + a^2d(x),
\]
since \(d(a^2)=2ad(a)\); or
\[
(4) \quad a^2d(x) = d(a)x + ad(x)a - axd(a) \quad \text{for all } a, x \in R.
\]
In (3), replace \(b\) by \(xa\), and find similarly
\[
(5) \quad d(x)a^2 = ad(x)a + axd(a) - d(a)xa, \quad \text{for all } a, x \in R.
\]
Add (4) and (5).
\[
(6) \quad a^2d(x) + d(x)a^2 = 2ad(x)a \quad \text{for all } a, x \in R,
\]
or
\[
(7) \quad a(d(x)a - ad(x)) = (d(x)a - ad(x))a \quad \text{for all } a, x \in R.
\]
Replace in (7) \(a\) by \(a+d(x)\); we find that \(d(x)\) commutes with \(d(x)a - ad(x)\), for all \(a \in R\), for all \(x \in R\); this says that the square of the inner derivation by \(x\) is zero, for all \(x \in R\). Let \(R\) not be of characteristic 2. Then Theorem 1 says that \(d(x)\) is central, for all \(x \in R\); let \(a\) be an element of \(R\), and \(A\) denote inner derivation by \(a\). \(ad(x)=d(x)a\), or \(A^d(x)=0\) for all \(x \in R\). Theorem 1 again shows that \(d=0\) or, if not, then \(A\) is zero, every \(a \in R\) is central, \(R\) is commutative. But if \(R\) is of characteristic 2, (6) says that for all \(x \in R\), \(d(x)\) commutes with all squares of elements of \(R\). Let \(R\) be a prime ring of characteristic 2, and let \(e \in R\) commute with \(a^2\), for all \(a \in R\).
\[
(8) \quad a^2e = ea^2 \quad \text{for all } a \in R.
\]
Replace \(a\) by \(a+b\) and use \(ea^2=a^2e\), \(eb^2=b^2e\).
\[
(9) \quad (ab + ba)e = e(ab + ba) \quad \text{for all } a, b \in R.
\]
In (9), replace \(b\) by \(ae\) and commute \(e\) and \(a^2\); then \(a^2e^2+aeae=ea^2e+eaea\); \(a^2e^2=ea^2e\), so
In (9), replace $b$ by $e$; then $a e^2 + e a e = e a e + e^2 a$,

(11) \[ e^2 \text{ is in the center of } R. \]

Consider $(a e + e a)^2 = e a e + e a e + a e^2 a + e a^2 e$. But $a e a e + e a e = 0$ by (10), $a e^2 a + e a^2 e = e^2 a^2 + e^2 a^2 = 0$ by (11) and (8). We have

(12) \[ (a e + e a)^2 = 0 \]

for all $a \in R$. Let $x, y$ now be elements of $R$ with $x y = 0$. By (9), $(x y + y x)e = e(x y + y x)$, so

(13) \[ x y = 0 \text{ implies } y x e = e y x. \]

Now $x^2 y = 0$, so (13) becomes also $y x^2 e = e y x^2$; $y x^2 e = y e x^2$ since $e$ commutes with all squares. Thus

(14) \[ x y = 0 \text{ implies } (y e + e y)x^2 = 0. \]

But $(a x) y = 0$ for all $a \in R$; then we can replace $x$ by $a x$ in (14), to obtain $(y e + e y)a x a x = 0$ for all $a \in R$, whenever $x y = 0$. Lemma 2 now says $x = 0$ or $y e + e y = 0$; in fact, since $x(y v) = 0$ for all $v \in R$, Lemma 2 even says $x = 0$ or $y v e + (e y)v = 0$ for all $v \in R$. Since $y e = e y$ if $x \neq 0$, then $x = 0$ or $y v e + y v e = 0$ for all $v \in R$, $y(v e + e v) = 0$ for all $v \in R$. Lemma 1 applied to the inner derivation by $e$ shows that either $x = 0$, $y = 0$, or $e$ is central. But by (12) $(a e + e a)(a e + e a) = 0$, for all $a \in R$; putting $x = a e + e a$, $y = a e + e a$, we find that for all $a \in R$, $a e + e a = 0$, or $e$ is central. That is, for all $a \in R$, $a e + e a = 0$, $e$ is central if $e$ commutes with all squares in $R$.

For all $x \in R$, then, $d(x)$ commutes with all squares in $R$, $d(x)$ is central for all $x \in R$. Let $d(b) = 0$; for all $a \in R$, $d(a b) = d(a) b + a d(b) = d(a) b$; $d(ab)$ is central, so $d(a) b$ is central for all $a$ in $R$ if $d(b) = 0$. Now if $d$ is not zero, so that $d(a) \neq 0$ for some $a \in R$, we have $d(a) b = x d(a) b$; $d(a)$ is central so $x d(a) b = d(a) x b$, whence $d(a) (b x + x b) = 0$ for all $x \in R$, if $d(b) = 0$. But as previously remarked, no nonzero element of the centroid of $R$ has nonzero kernel; since we are assuming $d(a) \neq 0$, and since $d(a)$ is central, we have proved that $b$ is central whenever $d(b) = 0$. But for all $c \in R$, $d(c^2) = d(c) c + c d(c) = 2 d(c) c = 0$, so $c^2$ commutes with all $x$ in $R$, for all $c \in R$. Referring back to the conclusion of the previous paragraph with $x$ for $e$ shows $x$ central for all $x \in R$, if $d$ is not the zero derivation.

The following lemma may also be of independent interest.

Lemma 4.2 Let $A$ be a Lie ring, $I$ an ideal of $A$, $d$ an element of $A$ such

$^2$ An oral communication from Professor Kaplansky.
that \(dx \cdot x = 0\) for all \(x \in I\). Then for all \(a \in R\), \((da \cdot x) x = 0\) for all \(x \in I\) (i.e. the set of \(d\) satisfying \(dx \cdot x = 0\) for all \(x \in I\) is an ideal of \(A\)).

**Proof.** Let \(R_a\) denote right multiplication by \(a\). We want to prove \(d(R_aR_x^2) = 0\) for all \(a \in A\), \(x \in I\). The Jacobi identity for a Lie ring may be written as \(R_{ax} = R_aR_x - R_xR_a\). Furthermore, since \(I\) is an ideal, it contains \(ax\), and \(x + ax\), for all \(a \in A\), so that \((d \cdot ax)ax = 0\), \((d(x + ax)) \cdot (x + ax) = 0\) for all \(a \in A\). From these two equations, and from \(dx \cdot x = 0\), we get \(dx \cdot ax + (d \cdot ax) \cdot x = 0\) for all \(a \in A\), \(x \in I\), or, in the other notation, \(d(R_xR_{ax} + R_{ax}R_x) = 0\). But from

\[
d(R_xR_{az} + R_{az}R_x) = d(R_x(R_aR_x - R_xR_a) + (R_aR_x - R_xR_a)R_x)
\]

\[
= d(R_xR_aR_x - R_x^2R_a + R_aR_x^2 - R_xR_aR_x) = d(R_xR_a^2 - R_x^2R_a),
\]

\(d(R_xR_a^2 - R_x^2R_a) = 0\) for all \(a \in A\), \(x \in I\). By hypothesis, \(d(R_x^2) = 0\), so that \(d(R_xR_a^2) = 0\) for all \(a \in A\), \(x \in I\). This is exactly what we had to prove.

We are now ready for Theorem 2.

**Theorem 2.** Let \(R\) be a prime ring and \(d\) a derivation of \(R\) such that, for all \(a \in R\), \(ad(a) - d(a)a\) is in the center of \(R\). Then, if \(d\) is not the zero derivation, \(R\) is commutative.

**Proof.** Let \(A\) be the Lie ring of derivations of \(R\) and \(I\) the ideal of \(A\) consisting of inner derivations. Let, for \(a \in R\), \(I_a\) denote inner derivation by \(a\). Let \([d_1, d_2]\) for \(d_1, d_2 \in A\) denote the (commutator) product of derivations in \(A\). We are assuming \([d, I_a], I_a] = 0\). By the preceding lemma, for all \(x \in R\), that is, for all \(x \in I\), \([[[d, I_x]I_a]I_x] = 0\) for all \(a \in R\). That is, \(a(ad(x) - d(x)a) - (ad(x) - d(x)a)a\) is central for all \(x, a \in R\),

\[(15) \quad a^2d(x) + d(x)a^2 - 2ad(x)a\] is central for all \(x, a \in R\).

Commuting (15) with \(a\),

\[(16) \quad 3ad(x)a^2 + a^3d(x) = 3a^2d(x)a + d(x)a^3.\]

Suppose \(R\) is of characteristic 3. Then for all \(a \in R\), \(I_3d = 0\). Theorem 1 says that \(d\) is zero, or else every \(a^3\) is in the center of \(R\); if this is the case, then for all \(a, b \in R\), \((a + b)^3 - a^3 - b^3 = a^2b + aba + ba^2 + b^2a + bab + ab^2\) is central; replace \(a\) by \(-a\) to find \(a^2b + aba + ba^2 - (b^2a + bab + ab^2)\) central for all \(a, b \in R\); adding these last two and dividing by 2, we see that \(a^2b + aba + ba^2\) is central, for all \(a, b \in R\). Replace \(b\) by \(ab\):

\(a^2b + a^2ba + aba^2 = a(a^2b + aba + ba^2)\) is central; if \(a^2b + aba + ba^2\) is not zero, given \(a\), for some \(b\), then, since it is central, we can divide by it, whence \(a\) would be central. So assume that \(R\) has the property that
for all \(a, b \in \mathbb{R}\), \(a^2b + aba + ba^2 = 0\). This reads, since \(\mathbb{R}\) is of characteristic 3, as \(a(ab - ba) - (ab - ba)a = 0\) for all \(b \in \mathbb{R}\), \(I_a^2 = 0\); by Theorem 1, \(a\) is central, \(\mathbb{R}\) is commutative.

Suppose now that \(\mathbb{R}\) is of characteristic different from 3. Write \(d(x) = x'\). In (16), replace \(x\) by \(a\): \(3aa'a^2 + a^2a' - 3a^2a'a - a'a^2 = 0\), or \(a'^3 = 3aa'a - 3a^2a^2 = 3(aa' - a'a)a\). Since \(aa' - a'a\) is central by the hypothesis of this theorem, we find

\[
(17) \quad a^3a' - a'a^3 = 3(aa' - a'a)a^2, \quad \text{for all } a \in \mathbb{R}.
\]

Furthermore, \((aa' - a'a)a = aa'a - a'a^2\). But \((aa' - a'a)a = a(aa' - a'a) = a^2a' - aa'a\); adding these last two, we obtain

\[
(18) \quad 2(aa' - a'a)a = a^2a' - a'a^2.
\]

In (16), replace \(x\) by \(ax'\).

\[
3a^2x''a^2 + a^4x'' - 3a^2x''a - ax''a^3 + 3aa'x'a^2 + a^3a'x' - 3a^2a'x'a - a'x'a^3 = 0.
\]

However,

\[
3a^2x''a^2 + a^4x'' - 3a^2x''a - ax''a^3 = a(3ax''a^2 + a^3x'' - 3a^2x''a - x''a^3) = 0,
\]

as is seen from (16) by replacing \(x\) by \(x'\). So

\[
(19) \quad 3aa'x'a^2 + a^3a'x' - 3a^2a'x'a - a'x'a^3 = 0, \quad \text{for all } x, a \in \mathbb{R}.
\]

Multiply (16) on the left by \(a'\).

\[
(20) \quad 3a'ax'a^2 + a'a^3x' - 3a'a^2x'a - a'x'a^3 = 0.
\]

Subtract (20) from (19):

\[
3(aa' - a'a)x'a^2 + (a^3a' - a'a^3)x' - 3(a^2a' - a'a^2)x'a = 0
\]

for all \(x, a \in \mathbb{R}\).

Using (17) and (18), we arrive at, after dividing by 3,

\[
(aa' - a'a)(x'a^2 + a^2x' - 2ax'a) = 0 \quad \text{for all } x, a \in \mathbb{R}.
\]

If \(aa' - a'a \neq 0\) for some \(a\), then for that \(a\), and all \(x\),

\[
(21) \quad x'a^2 + a^2x' - 2ax'a = 0.
\]

Replace \(x\) by \(ax\) in (21):

\[
a^x'a^2 + a^2x' - 2a^2x'a + a'xa^2 + a^2a'x - 2a'a^xa = 0;
\]
since

\[ ax'a^2 + a^3x' - 2a^2x'a = a(x'a^2 + a^2x' - 2ax'a) = 0 \]

by (21), we have

(22) \[ a'xa^2 + a^2a'x - 2aa'xa = 0 \]

for all \( x \in R \).

Now in (21) replace \( x \) by \( a: a'a^2+a^2a'-2aa'a=0 \). Multiply this on the right by \( x \).

(23) \[ a'a^2x + a^2a'x - 2aa'ax = 0 \]

for all \( x \in R \).

Subtract (23) from (22).

(24) \[ a'(xa^2 - a^2x) - 2aa'(xa - ax) = 0 \]

for all \( x \in R \).

Replace \( x \) by \( ax \) in (24).

(25) \[ a'(xa^2 - a^2x) - 2aa'(xa - ax) = 0 \]

for all \( x \in R \).

Multiply (24) by \( a \) on the left.

(26) \[ aa'(xa^2 - a^2x) - 2a^2a'(xa - ax) = 0 \]

for all \( x \in R \).

Subtract now (25) from (26):

(27) \[ (aa' - a'a)(xa^2 - a^2x) - 2a(aa' - a'a)(xa - ax) = 0 \]

for all \( x \in R \).

Since \( aa' - a'a \neq 0 \),

(27) \[ xa^2 - a^2x - 2a(xa - ax) = 0 \]

for all \( x \in R \) if \( aa' - a'a \neq 0 \).

So \( xa^2 + a^2x - 2axa = 0 \), \( a(ax - xa) = (ax - xa)a \), \( I_a^2 = 0 \). That is, \( a \) is central by Theorem 1 or else \( aa' = a'a \), if \( R \) is of characteristic different from 2. So when \( R \) is of characteristic not 2, \( aa' = a'a \) for all \( a \in R \); Lemma 3 now finishes the proof. Let \( R \) finally be of characteristic 2. (27) says \( aa' = a'a \) or else \( a^2 \) is central, for all \( a \in R \). If \( aa' = a'a \), for some \( a \in R \), \( a^2 \) is central and not zero. For if \( a^2 = 0 \) then \( (a^2)' = aa' + a'a = 0 \), \( aa' = a'a \). Then \( a \) is not a divisor of zero, since if \( xa = 0 \), \( ya^2 = 0 \), \( y = 0 \). Let \( x \in R \); we shall prove that \( aa' \) commutes with \( x^2 \). Either \( (axa)^2 \) is central, or \( (axa)(axa)' = (axa)'(axa) \). If \( (axa)^2 \) is central, \( axa^2xa \) is in the center of \( R \). Then \( ax^2a \) is in the center of \( R \), since \( a^2 \) is; call it \( c \). Then \( acc = a^2c \) is in the center of \( R \), and equals \( a^2x^2a^2 \). So \( a^2x^2a^2 \) is in the center of \( R \), and so is \( x^2 \), whence \( x^2 \) commutes with \( aa' \) if \( (axa)^2 \) is central. On the other hand, if \( x^2 \) is not central, then \( xx' = x'x \) and \( (axa)(axa)' = (axa)'(axa) \). Then \( (axa)'(a'xa + ax'a + ax'a') = (a'xa + ax'a + ax'a')axa \), or

\[ axaa'xa + axa^2xa' + axa^2xa' = a'xa^2xa + axa^2xa + axa'axa. \]
Now $a^2$ is central, whence
\[ ax(aa' + a'a)x + (a(xx' + x'x)a + ax^2a' + a'x^2a)a^2 = 0. \]
But $xx' + x'x = 0$, and $aa' + a'a$ is central so that
\[ (aa' + a'a)ax^2a + (ax^2a' + a'x^2a)a^2 = 0. \]
Since $a$ is not a right zero divisor,
\[ (aa' + a'a)ax^2 + (ax^2a' + a'x^2a)a = 0, \]
\[ ax^2(aa' + a'a) + (ax^2a' + a'x^2a)a = 0, \]
\[ ax^2a^2 + ax^2a'a + ax^2a'a + a'x^2a^2 = 0. \]
Thus $ax^2a + a'x^2a^2 = 0$; $a^2$ is central so $ax^2aa' + a^2a'x^2 = 0$; $a$ is not a left divisor of zero so $x^2aa' + aa'x^2 = 0$, for any $x$ such that $x^2$ is not central, hence, for all $x \in R$, as promised; otherwise $aa' = a'a$. Recourse to the latter part of Lemma 3 shows $a^2$ central and $aa'$ central or else $aa' = a'a$. But in the former case, $a \cdot aa' = aa' \cdot a$; since $a$ is not a zero divisor, $aa' = a'a$, for all $a \in R$. Lemma 3 completes the proof.

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