APPROXIMATION BY THE TRANSLATES OF A SINGLE FUNCTION

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In 1941 Seidel and Walsh [3] proved the existence of an entire function F of a complex variable such that every function analytic in a simply connected region of the complex plane is the uniform limit on compact sets of a sequence of translates of F. This result generalizes a theorem of G. D. Birkhoff [1] on entire functions. In the present note an analogous theorem is proved for continuous real or complex functions on a more general class of topological spaces where the role of polynomial approximation in the above proofs is assumed by a sequence of functions constructed using Urysohn's lemma.

Let X be a locally compact hausdorff space with the following properties: there exist countable sequences $\{C_n\}$ and $\{\sigma_n\}$ of disjoint compact sets and homeomorphisms of X onto itself, respectively, such that for every compact K, Ia. $K \cap C_n = \emptyset$ and Ib. $K \subset C_n\sigma_n$,[1] except for finitely many n. Such an X is evidently not compact but is countable at infinity, since each point lies in some $C_n\sigma_n$. Thus the compact open topology on the algebra $\mathfrak{A}$ of all continuous real or complex valued functions on X is the topology of sequential convergence in a suitable Fréchet metric on $\mathfrak{A}$.

THEOREM. *Let X be a locally compact hausdorff space with properties* Ia *and* Ib, *and let $\mathfrak{Y}$ be a countable family of continuous real or complex functions on X. Then there exists a continuous real or complex function F on X such that every uniform limit on compact sets of functions in $\mathfrak{Y}$ is the limit of a sequence of the functions $F \circ \sigma_n^{-1}$.*

First we find an infinite subsequence $\{C_m\}$ of $\{C_n\}$ and sets Δ_m and W_m, compact and open respectively, such that $\Delta_m \subset W_m \subset C_m$ and such that $\{\Delta_m\sigma_m\}$ retains property Ib. Since X is locally compact, the interiors U_n of $C_n\sigma_n$ are nonempty for an infinite set J of integers, and $\{U_j, j \in J\}$ has property Ib. In fact, each compact K has a compact neighborhood N, and $N \subset C_j\sigma_j$ for all j large; thus the interior of N, which contains K, is contained in U_j. The $C_j\sigma_j$, $j \in J$, are compact, so that for each j there is a least integer $\gamma(j) \in J$ such that $C_j\sigma_j \subset U_{\gamma(j)}$. For each m in the range M of the function γ choose j such that $\gamma(j) = m$ and define $\Delta_m = \overline{U}_j\sigma_m^{-1}$ and $W_m = U_m\sigma_m^{-1}$. If j is not in the range

Received by the editors March 5, 1957.

[1] We write $C_n\sigma_n$ instead of $\sigma_n(C_n)$. All indices are positive integers.

of γ, then $C_j\sigma_j \subset U_{\gamma(j)}$, and $\{C_m\sigma_m, m \in \mathrm{M}\}$ retains property Ib. Thus $\{\Delta_m\sigma_m\}$ has property Ib also. Evidently $\Delta_m \subset W_m \subset C_m$, since $\overline{U}_j \subset U_m \subset C_m\sigma_m$. Reindex the Δ's, W's, and C's, using all the positive integers.

Next we construct compact sets Γ_m with the properties IIa. $\Gamma_m \subset \Gamma_{m+1}$; $\bigcup \Gamma_m = X$; every compact $K \subset \Gamma_m$, if m is large. And IIb. $\bigcup_{i=1}^m \Delta_i \subset \Gamma_m$ and $\Gamma_m \cap \Delta_{m+1} = \varnothing$. Define Γ_m as

$$\bigcup_{i=1}^{m} \overline{W}_i \cup \left[\overline{U}_m \cap C\left(\bigcup_{m+1}^{\infty} W_i\right)\right].$$

Now $\overline{U}_m \subset \overline{U}_{m+1}$, and $\mathbf{C}(\bigcup_{m+1}^{\infty} W_i) \subset \mathbf{C}(\bigcup_{m+2}^{\infty} W_i)$; thus $\Gamma_m \subset \Gamma_{m+1}$. Since $\overline{U}_m$ is compact, so is Γ_m. Further

$$\overline{U}_m \subset \left[\overline{U}_m \cap \mathbf{C}\left(\bigcup_{m+1}^{\infty} W_i\right)\right] \cup \bigcup_{1}^{\infty} W_i \subset \bigcup_{1}^{\infty} \Gamma_m, \text{ and } \bigcup \Gamma_m = X.$$

Since $W_m \subset C_m$, each compact K meets only finitely many W_m and lies in all but finitely many $\overline{U}_m$; thus $K \subset \Gamma_m$ for all m large. The first part of IIb is trivial. For the second part, observe that $\overline{W}_m \subset C_m$ and $C_m \cap \Delta_{m+1} = \varnothing$. Also $[\overline{U}_m \cap \mathbf{C}(\bigcup_{m+1} W_i)] \subset \mathbf{C} W_{m+1}$ and $\Delta_{m+1} \subset W_{m+1}$. Thus $\Gamma_m \cap \Delta_{m+1} = \varnothing$.

We are now in a position to construct F. Let $\{f_m\}$ be the family $\mathfrak{Y}$ indexed by the positive integers, in such a way that each function is repeated countably often, and construct continuous functions α_m, β_m, and g_m as follows, using Urysohn's lemma:

$$\alpha_m(x) = \begin{cases} 0 \text{ on } \Delta_m, \\ 1 \text{ on } \Gamma_{m-1}, \end{cases} \qquad \beta_m(x) = \begin{cases} 1 \text{ on } \Delta_m, \\ 0 \text{ on } \Gamma_{m-1}, \end{cases}$$

$$\begin{cases} g_1(x) = f_1(x), \\ g_m(x) = \alpha_m(x) g_{m-1}(x) + \beta_m(x) f_m(x\sigma_m). \end{cases}$$

Observe that $g_m(x) = g_{m-1}(x)$ on Γ_{m-1} and that $g_m(x) = f_m(x\sigma_m)$ on Δ_m. Since each compact K lies in all Γ_m from some m on, the sequence $\{g_m\}$ converges uniformly on compact sets to a limit F; this function is continuous, since it coincides with a continuous function on each compact set, and the space X is locally compact. Now $F(x) = g_m(x)$ on $\Gamma_m \supset \Delta_m$, so that $F(x) = f_m(x\sigma_m)$ on Δ_m. Let $y \in \Delta_m\sigma_m$ and write $y = x\sigma_m$ for some $x \in \Delta_m$. Then $F(y\sigma_m^{-1}) = F(x) = f_m(x\sigma_m) = f_m(y)$ for $y \in \Delta_m\sigma_m$. The sequence $\{\Delta_m\sigma_m\}$ has property Ib; so suppose that the sequence $\{f_{n_i}\}$ of functions from $\mathfrak{Y}$ converges uniformly on compact sets to a function f. For each compact K there is an i_0 such that $K \subset \Delta_{n_i}\sigma_{n_i}$

for $i \geqq i_0$.[2] But $f_{n_i}(y) = F(y\sigma_{n_i}^{-1})$ for $y \in \Delta_{n_i}\sigma_{n_i}$, $i \geqq i_0$. Thus $\{F(y\sigma_{n_i}^{-1})\}$ converges uniformly to f on K.

COROLLARY 1. *The theorem can be proved under hypothesis* Ib *and the following condition*: Ia′. *There exist open sets $V_n \supset C_n$ such that $V_n \cap C_m = \emptyset$, if $n \neq m$, and the set $\bigcup C_n$ is closed.*

It is enough to show that for every compact K, $K \cap C_n = \emptyset$, except for finitely many n. Assume that $K \cap C_m \neq \emptyset$ for an infinite subset M of $\{n\}$, and choose $p_m \in K \cap C_m$ for each $m \in$ M. Then $\{p_m\}$ is an infinite point set, which must have a limit point p in $K \cap \mathrm{cl}\,(\bigcup C_m)$. But if $p \in \mathrm{cl}\,(\bigcup C_m)$, then $p \in \bigcup C_n$, since $\bigcup C_n$ is closed, and so $p \in C_r$ for some r. Thus infinitely many p_m lie in V_r, which is impossible for $m \neq r$; consequently $K \cap C_n = \emptyset$, for all n large.

COROLLARY 2. *If X is a differentiable manifold of class r, $1 \leqq r \leqq \infty$, and if the σ_n and the f_n are of class r, then F can be found also of class r.*

The functions α_m and β_m can be chosen of class r,[3] so that the g_n are also of class r. For each point $p \in X$, choose a compact neighborhood N. Then for some m, $N \subset \Gamma_m$; on Γ_m the function F equals g_m, which is of class r. Thus F has class r.

REFERENCES

1. G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, Collected Papers, vol. III, pp. 307–309.

2. G. de Rham, *Variétés différentiables*, Paris, 1955.

3. W. Seidel and J. L. Walsh, *On approximation by euclidean and non-euclidean translations of an analytic function*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 916–920.

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[2] If $n_i = m_0$ for infinitely many i, then $\lim f_{n_i} = f_{m_0}$. Since f_{m_0} occurs countably often among the f_m, it is the limit of translates of F.

[3] See for example [2, p. 6].