A FIXED POINT THEOREM

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1. The fixed point theorem. Let \( T: X \to Y \) be a point-to-set function and let \( T^{-1}: Y \to X \) be the point-to-set function such that \( x \in T^{-1}(y) \) if and only if \( y \in T(x) \). If \( X \) and \( Y \) are topological spaces, \( T \) is continuous provided the function \( T^{-1} \) is both open and closed, and each set \( T(x) \) is closed.

Let \( H \) be Čech homology theory with coefficients in a field, and let \( E_n \) be Euclidean \( n \)-space. A mapping \( f: B \to E_n \) will be said to link a point of \( E_n \) provided that \( x \in f(B) \) and the induced homomorphism \( f_*: H_{n-1}(B) \to H_{n-1}(E_n - x) \) is not zero.

Theorem. Let \( C \) be a compact connected subspace of \( E_n \) and let \( T: C \to C \) be a continuous point-to-set function. Let \( B \) be a compact space, \( I \) the unit interval, with \( h: B \times I \to E_n \) a homotopy such that \( h_0 \) links every point of \( C \) and \( h_1 \) links no point of \( C \). Then there exists a number \( t \) with \( 0 < t \leq 1 \) and a point \( y \in C \) such that \( y \in h_t(B) \) and \( h_t(B) \) meets \( T(y) \).

(Here, as usual, \( h_t \) denotes the map of \( B \) into \( E_n \) such that \( h_t(b) = h(b, t) \)). We note that the hypothesis on \( h \) is fulfilled in case \( B \) is the boundary of a bounded open set containing \( C \) and \( h \) is a deformation of \( B \) such that \( E_n - h_1(B) \) is connected.

To prove the theorem we need the following lemmas.

Lemma 1 (Notation as in Theorem). In the space \( C \times E_1 \), let \( U = \{ (x, t) \mid t < 0 \} \), \( V = \{ (x, t) \mid t > 1 \} \), \( B^* = \{ (x, t) \mid x = h(b, t) \text{ for some } b \in B \} \). Then \( U \) and \( V \) are contained in different components of \( C \times E_1 - B^* \).

Proof. Let \( L \) be the set of points \( (x, t) \) of \( C \times I \) such that \( h_t \) links \( x \). In view of the hypotheses on \( h_0 \) and \( h_1 \), it suffices to prove that \( L \) and \( C \times I - (B^* \cup L) \) are open in \( C \times I \). If \( (x, t) \in C \times I - B^* \), there is an \( \varepsilon \)-neighborhood \( N \) of \( x \) in \( E_n \) and a neighborhood \( P \) of \( t \) in \( I \) such that if \( u \in P \) then \( h_u(B) \cap N = \emptyset \). Consider the neighborhood \( (N \cap C) \times P \) of \( (x, t) \) in \( C \times I \). If \( (y, u) \) is in this neighborhood, then \( h_t \) and \( h_u \) are homotopic, considered as maps into \( E_n - N \). But the inclusion maps of \( E_n - N \) into \( E_n - x \) and \( E_n - y \) induce homology isomorphisms. Thus \( h_t \) links \( x \) if and only if \( h_u \) links \( y \), and the proof is complete.

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A subset \( A \) of a space \( X \) will be said to separate two points \( a \) and \( b \) of \( X \) provided every compact connected subset \( F \) of \( X \) containing \( a \) and \( b \) meets \( A \).

**Lemma 2.** Let the points \( a \) and \( b \) of a compact space \( X \) be separated by a closed subset \( A \) of \( X \). If the homology group \( H_1(X) \) is zero, then there is a compact connected subset \( K \) of \( A \) which separates \( a \) and \( b \).

**Proof.** An application of Zorn's lemma shows that there is a minimal compact subset \( K \) of \( A \) which separates \( a \) and \( b \). But \( K \) is connected, for if not \( K \) can be expressed as the union of two disjoint nonempty closed subsets of \( X \) neither of which separates \( a \) and \( b \). However this would contradict Theorem VII 9.2 of [1].

**Lemma 3.** If \( T: X \to Y \) is a continuous point-to-set function from a compact connected space \( X \) onto a compact space \( Y \) and if \( L \) is a component of \( Y \), then \( T^{-1}(L) = X \).

**Proof.** The set \( L \) is the intersection of its open-and-closed neighborhoods \( N_\alpha \), and since \( T \) is continuous and \( X \) connected, \( T^{-1}(N_\alpha) = X \) for each \( \alpha \). Thus the closed sets \( T(x) \cap N_\alpha \) are all nonempty, and for a particular \( x \in X \) the collection \( \{ T(x) \cap N_\alpha \} \) has the finite intersection property, so that \( T(x) \) meets \( L \).

**Proof of the theorem.** Identify the points \((x, t)\) of \( C \times E_1 \) for which \( t \geq 2 \) (call this point \( \alpha \)) and the points for which \( t \leq -1 \) (call this point \( \beta \)). The resulting space \( \tilde{C} \) is the two-point suspension of \( C \). Using the Mayer-Vietoris sequence [2] one finds that since \( C \) is connected, \( H_1(\tilde{C}) = 0 \). By Lemma 1, \( B^* \) separates \( \alpha \) and \( \beta \) in \( \tilde{C} \), and thus, by Lemma 2, there is a compact connected subset \( K \) of \( B^* \) which also separates \( \alpha \) and \( \beta \). The function \( T \times 1: C \times E_1 \to C \times E_1 \) is continuous and determines in a natural way a continuous function \( \tilde{T}: \tilde{C} \to \tilde{C} \). Let \( L \) be a component of the compact space \( T(K) \). Suppose \( L \) and \( K \) are disjoint. By Lemma 3 \( T^{-1}(L) = K \); hence in particular \( K \) and \( L \) have the same projections on \( E_1 \). Thus there are arcs in \( \tilde{C} - K \) from \( \alpha \) and \( \beta \) to \( L \). But this contradicts the fact that \( K \) separates \( \alpha \) and \( \beta \), hence \( L \) meets \( K \). Thus there is a \( t, 0 < t \leq 1 \), such that \( T(h_t(B)) \) meets \( h_t(B) \).

Note that the theorem remains true if \( T \) is an upper semi-continuous function such that \( T(x) \) is connected for each \( x \), for then \( \tilde{T}(K) \) is itself connected and can replace \( L \) in the preceding proof.

**2. Applications.**

1. Let \( B \) be an \( n \)-sphere in \( E_{n+1} \) and \( C \) a (concentric) \( n \)-sphere in the interior of \( B \). Let \( d \leq \text{diam} \ C \) and let \( h_t \) be as in the Theorem. Then some \( B_t = h_t(B) \) will intersect \( C \) in points \( x, y \) with \( d(x, y) = d \).
Proof. Let $C_x$ be the set of points on $C$ whose distance from $x$ is $d$. Then the mapping $T(x) = C_x$ is continuous.

Remark. We can define the function $t(x)$ on $C$ as the "time" $t$ at which $x \in B_t$. This function is not necessarily single valued and therefore usually upper semi-continuous only but it is a function with connected graph. Thus the theorem of Kakutani-Yamabe-Yujobo [3] can be generalized for this function to yield that for some $t$ the set $B_t \cap C$ contains the endpoints of $n+1$ mutually orthogonal radii of $C$.

2. W. Gustin has raised the following question. Given a convex surface $C$ in $E_3$ what is the minimal length of a closed curve that can be "slipped over" $C$? He remarked that the minimal perimeter of all orthogonal projections of $C$ is obviously long enough to be slipped over $C$ even without bending. I. Schoenberg has shown that this length is not minimal even for tetrahedra and conjectured that the minimal length is that of the minimal closed geodesic on $C$. There are good heuristic arguments in favor of this conjecture, but it has not been proved completely so far. 2

With the help of our theorem we can obtain some information on the problem even without the assumption that $C$ is convex.

The process of "slipping over" can be replaced by a varying simple closed curve $K_t$ ($0 \leq t \leq 1$) on $C$ whose closed interior expands from a point for $K_0$ to the whole of $C$ for $K_1$. This process can in turn be replaced by the shrinking of a surface $B$ which contains $C$ in its interior to a point in the interior of $C$, if we replace the interior of $K_t$ on $C$ by its image under a slight radial contraction towards a point $P$ in the interior of $C$, and the exterior of $K_t$ on $C$ by its image under a slight radial dilation from $P$. We then connect the two pieces by the necessary part of the cone through $K_t$ with vertex $P$.

Thus we can rephrase the problem. Let $B$ be a closed surface containing the convex surface $C$. Let $B$ be shrunk to a point so that $B_t \cap C$ is a rectifiable simple curve $K_t$, and let $l$ be the maximum of the length of $K_t$. Then what is the minimum of $l$ for all possible contractions $B_t$?

From our theorem we know that for every continuous transformation $T$ of $C$ into itself, $K_t$ will have to pass through a point $x$ and intersect $T(x)$ for some $t$. Thus the mapping into diametrically opposite points proves Gustin's curves to be minimal for spheres and some other centrally symmetric surfaces (such as right circular cylinders and ellipsoids).

2 Added in proof. Schoenberg's conjecture has been proved by H. Busemann and will appear in his book *Convex surfaces*, Interscience Publishers, New York.
We may obviously restrict our attention to arbitrarily smooth surfaces $C$. For such surfaces we can define the continuous mapping $T$ which maps every point $x$ into the set of points whose geodesic distance from $x$ is no less than that of the nearest conjugate point of $x$. Thus we obtain that $l$ is no less than twice the distance between the nearest conjugate points on $C$.

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**References**