ON THE BOUNDEDNESS OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1. In this note we shall derive some criteria for the boundedness of solutions of nonlinear differential equations by means of a simple lemma given below. We shall also consider when a solution of a differential equation is unbounded.

We shall say that the function $h(x, r)$ possesses the property I if $h(x, r) \geq 0$ for the specified range of values of $x$ and $r$, if it is measurable in $x$ for fixed $r \geq 0$, continuous in $r$ for fixed $x$, $x_0 \leq x < \infty$, $r \geq 0$, and if $r(x)$ is the maximal solution of the differential equation $r' = h(x, r)$ passing through the point $(x_0, 0)$.

Let us begin by proving the following lemma:

**Lemma.** Suppose that $h(x, r)$ has property I. Let $y(x)$ be continuous on $x_0 \leq x < \infty$ and satisfy the inequality $|\Delta y(x)| \leq \int_{x_0}^x \Delta z h(t, y(t)) dt$, $\Delta x \geq 0$, then $y(x) \leq r(x)$ for $x_0 \leq x < \infty$.

**Proof.** The inequality shows that $y(x)$ is absolutely continuous in the interval $[x_0, \infty)$, which implies that the derivative $y'(x)$ exists almost everywhere in $[x_0, \infty)$. Furthermore, it is clear from the assumed inequality that the derivative satisfies the relation

$$|y'(x)| \leq h(x, y(x)),$$

almost everywhere.

Suppose that $b(x, \epsilon)$ is a solution of $r' = h(x, r) + \epsilon$, $r(x_0) = 0$ where $\epsilon$ is an arbitrarily small quantity. It is easy to show that

$$y(x) \leq b(x, \epsilon), \quad x_0 \leq x < \infty.$$

For suppose that this relation does not hold. Then, without loss of generality, let $[x_0, x_1]$ be an interval where $y(x) \geq b(x, \epsilon)$. At $x_0$, we have $y(x_0) = b(x_0, \epsilon)$. Hence taking right-hand derivatives at $x_0$, we obtain the inequality

$$y'(x_0) \geq b'(x_0, \epsilon).$$

From this we obtain the further inequality

$$h(x_0, y(x_0)) \geq h(x_0, b(x_0, \epsilon)) + \epsilon,$$

which leads to a contradiction. Hence (2) holds.

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Since we know that \( \lim_{\epsilon \to 0} b(x, \epsilon) = r(x) \), see [1], the lemma is proved.

Note. The notion of the maximal solution, and the above argument as used throughout this note, follow Kamke [1]. The lemma is a generalization of Bellman’s lemma, cf. [2].

2. Let \( y \) and \( f(x, y) \) be vectors with real components, \((y_1, y_2, \cdots, y_n)\) and \((f_1(x, y), f_2(x, y), \cdots, f_n(x, y))\) respectively. Define the norm of the vector \( y \) as follows: \( \|y\| = \sum_{i=1}^{n} |y_i| \). Consider the system
\[
y' = f(x, y), \quad y(x_0) = 0,
\]
where \( f(x, y) \) is continuous on \( x_0 \leq x < \infty \), \( \|y\| < \infty \).

Then we have

**Theorem 1.** Suppose that \( h(x, r) \) has property I and that
\[
\|f(x, y)\| \leq h(x, \|y\|).
\]
Then if \( r(x) = O(1) \) as \( x \to \infty \), the norm of every solution of (1) tends to a finite limit as \( x \to \infty \). If, in particular, \( r(x) = o(1) \) then each component of every solution of (1) tends to zero as \( x \to \infty \).

**Proof.** Let a solution of (1) be \( y(x) = \int_{x_0}^{x} f(t, y(t)) \, dt \), and let \( \Delta y(x) = y(x + \Delta x) - y(x) \), for \( \Delta x > 0 \). It follows that
\[
\|\Delta y(x)\| \leq \int_{x}^{x+\Delta} \|f(t, y(t))\| \, dt
\]
and hence that
\[
\|\Delta y(x)\| \leq \int_{x}^{x+\Delta} h(t, \|y(t)\|) \, dt.
\]
Using the lemma derived above, we obtain
\[
\|y(x)\| \leq r(x), \quad 0 \leq x < \infty.
\]
This together with the assumptions of the theorem yield the stated results, which generalize a result due to Wintner, [4].

The first part of the theorem is contained in [5] for the case where \( h(x, r) \) is monotone nondecreasing in \( r \). Here, we have merely assumed that \( h(x, r) \) is continuous.

**Theorem 2.** Suppose that \( h(x, r) \) has the property I. Let the differential system in (1) satisfy the further condition that
\[
\|f(x, y) - f(x, z)\| \leq h(x, \|y - z\|).
\]
Suppose that \( r(x) = O(1) \) as \( x \to \infty \). Then if one solution of (1) tends to
a finite limit as $x \to \infty$, then every solution vector tends to a finite limit. If, in particular, $r(x) = o(1)$ as $x \to \infty$, then every solution tends to the same finite limit as $x \to \infty$.

**Proof.** Let $y$ and $z$ be any two solutions of (1). Writing $v = y - z$ and proceeding as in the proof of the previous theorem, we obtain

\[
\| \Delta v \| \leq \int_{x}^{x+\Delta x} \| f(t, y) - f(t, z) \| dt
\]

and, as a consequence,

\[
\| \Delta v \| \leq \int_{x}^{x+\Delta x} h(t, \|v\|) dt, \quad \Delta x > 0.
\]

Applying the lemma proved above, this yields

\[
\|v\| = \|y - z\| \leq r(x), \quad x_0 \leq x < \infty.
\]

This together with the hypothesis of the theorem furnishes the desired result, which constitutes a generalization of a result of Wintner, [3]. When $n = 1$, the first part of the theorem is contained in [5].

**Corollary.** If in the above theorem we suppose that $f(x, 0) = 0$, then every solution tends to a finite limit as $x \to \infty$.

Let us now consider when a solution of the differential equation can be unbounded as $x \to \infty$. We treat only the case $n = 1$.

**Theorem 3.** Let $h(x, r) > 0$ be continuous on $x_0 < x < \infty$, $r > 0$ and $h(x, 0) = 0$. Suppose that the following condition is satisfied:

\[
| f(x, y_2) - f(x, y_1) | \geq h(x, | y_2 - y_1 |).
\]

Then, if any one solution of $r' = h(x, r)$ passing through $(x_0, 0)$ is unbounded as $x \to \infty$, then at least one solution of (1) is unbounded.

**Proof.** Let $y_1$ and $y_2$ be any two solutions of (1). Put $z = y_2 - y_1$. Consider the equation

\[
z' = y' - y' = f(x, y_2) - f(x, y_1)
\]

\[
= f(x, z + y_1) - f(x, y_1) = F(x, z).
\]

Using the assumption made above, we have $| F(x, z) | \geq h(x, | z |)$. Since $F(x, z)$ is continuous and $h(x, r) > 0$, the above inequality shows that, for ordinates different from zero, $F(x, z) \geq 0$; which implies that either $F(x, z) \geq h(x, | z |)$ or $F(x, z) \leq - h(x, | z |)$. Let us consider the first case.

It may be shown by means of an argument similar to that given
in the proof of the Lemma that \( z(x, \epsilon) \geq k(x) \) for \( x_0 \leq x < \infty \), where \( k(x) \) is the given unbounded solution and \( z(x, \epsilon) \) is a solution of \( z' = F(x, z) + \epsilon \) passing through \((x_0, 0)\).

It follows that

\[
(12) \quad z(x) = \lim_{\epsilon \to 0} z(x, \epsilon) \geq k(x), \quad x_0 \leq x < \infty,
\]

and that \( z(x) \) is a solution of \( z' = F(x, z) \) passing through \((x_0, 0)\). It follows that \( z(x) \) is the required unbounded solution. The proof of the other case is similar.

3. In this section, we shall compare the solutions of two different systems. Let \( z \) and \( g(x, z) \) be vectors with real components \((z_1, z_2, \cdots, z_n), (g_1(x, z), g_2(x, z), \cdots, g_n(x, z))\), and let the norm be defined as above. Consider the system

\[
(1) \quad z' = g(x, z), \quad z(x_0) = 0,
\]

where \( g(x, z) \) is continuous on \( x_0 \leq x < \infty, |z| < \infty \).

**Theorem 4.** Suppose that \( h(x, r) \) has property I. Let the functions \( f(x, y) \) and \( g(x, z) \) satisfy the condition

\[
(2) \quad \|f(x, y) - g(x, z)\| \leq h(x, |y - z|),
\]

and suppose that \( r(x) = O(1) \) as \( x \to \infty \).

Then, if one solution of \((3.1)\) tends to a finite limit as \( x \to \infty \), then every solution of \((2.1)\) tends to a finite limit as \( x \to \infty \), and conversely. If, in particular, \( r(x) = o(1) \) as \( x \to \infty \), then every solution of \((2.1)\) and \((3.1)\) tends to the same finite limit as \( x \to \infty \).

**Proof.** Suppose that \( y \) is a solution of \((2.1)\) and that \( z \) is a solution of \((3.1)\). Set \( v = y - z \). Proceeding as above, it follows that

\[
(3) \quad \|\Delta v\| \leq \int_x^{x+\Delta x} h(t, \|v\|)dt, \quad \Delta x > 0.
\]

Our lemma yields

\[
(4) \quad \|v\| \leq r(x), \quad x_0 \leq x < \infty.
\]

From this and our hypotheses, the desired conclusion follows.

**Corollary.** If \( g(x, 0) = 0 \) and \( r(x) = O(1) \) as \( x \to \infty \), then every solution of \((2.1)\) tends to a finite limit as \( x \to \infty \).

**Note.** A comparison theorem of this type is given in [5] for the case where \( n = 1 \) and \( h(x, r) \) is monotonically nondecreasing in \( r \).
Let us now consider another result concerning the unboundedness of a solution of a differential equation.

**Theorem 5.** Let \( h(x, r) > 0 \) be continuous for \( x_0 \leq x < \infty, \ r > 0, \) \( h(x, 0) = 0, \) and suppose that

\[
|f(x, y) - g(x, z)| \geq h(x, |y - z|) .
\]

Let \( g(x, 0) \equiv 0. \) Then the unboundedness of any one solution of \( r' = h(x, r) \) through \( (x_0, 0) \), apart from the minimal solution, guarantees the unboundedness of the maximal solution of (2.1) as \( x \to \infty. \)

**Proof.** Since \( g(x, 0) = 0, \) we have by virtue of (5),

\[
|f(x, y)| \geq h(x, |y|).
\]

By the argument similar to that of Theorem 3, this implies either \( f(x, y) \geq h(x, |y|) \) or \( f(x, y) \leq -h(x, |y|) \). Let us consider the first case. Proceeding as above, we obtain the inequality

\[
y(x, \epsilon) \geq k(x), \quad x_0 \leq x < \infty,
\]

where \( k(x) \) is the unbounded solution of \( r' = h(x, r) \) and \( y(x, \epsilon) \) is any solution of \( y' = f(x, y) + \epsilon \) passing through \( (x_0, 0) \). Since it is known that \( \lim_{x \to \infty} y(x, \epsilon) = z(x) \), the maximal solution of \( z' = f(x, z) \) passing through \( (x_0, 0) \), the conclusion follows from (7).

**Corollary.** If the minimal solution of \( r' = h(x, r) \) is unbounded, then every solution of (2.1) is unbounded as \( x \to \infty. \)

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**References**

4. ———, *An abelian lemma concerning asymptotic equilibria*, ibid.