ON COHOMOLOGY OF LIE ALGEBRAS

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Let G be a Lie algebra, L an ideal of G and M a G-module. The n-dimensional cochains for L in M are n-linear alternating functions on L^n with values in M. These cochains form a vector space $C^n(L, M)$ over the ground field of L; we identify $C^0(L, M)$ with M. $C^0(L, M)$ is already a G-module and for $n>0$, f in $C^n(L, M)$ and γ in G put

$$(\gamma\cdot f)(\sigma_1, \cdots, \sigma_n)$$
$$= \gamma\cdot f(\sigma_1, \cdots, \sigma_n) - \sum_{i=1}^{n} f(\sigma_1, \cdots, \sigma_{i-1}, [\gamma, \sigma_i], \sigma_{i+1}, \cdots, \sigma_n)$$

for all $\sigma_1, \cdots, \sigma_n$ in L.

This definition makes $C^n(L, M)$ into a G-module and further $\gamma\cdot(\delta f)=\delta(\gamma\cdot f)$ where δ is the coboundary operator: $C^n(L, M) \rightarrow C^{n+1}(L, M)$ so that the cohomology groups $H^n(L, M)$ take on the structure of a G-module. Professor Hochschild suggested to the author that he define an operation of G on the standard interpretations of the low dimensional cohomology groups which would agree with the operation of G on $H^n(L, M)$ as given above.

This problem has an analogue in the cohomology of groups. There, if L is a normal subgroup of a group G and f is any function of L^{n+1} to M, the operation

$$(\gamma\cdot f)(\sigma_0, \cdots, \sigma_n) = \gamma\cdot f(\gamma^{-1}\sigma_0\gamma, \cdots, \gamma^{-1}\sigma_n\gamma)$$

for γ in G and $\sigma_0, \cdots, \sigma_n$ in L, again induces the structure of a G-module on the cohomology groups $H^n(L, M)$. In an appendix we outline briefly a way of making the standard interpretations of the low dimensional cohomology groups into G-modules in a way consistent with the operation of G on the cohomology groups.

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1. Let G be a Lie algebra over a ground field of *characteristic* 0, L an ideal of G such that G/L is semi-simple and let M be a G-module. We denote by ρ_n the restriction homomorphism of $H^n(G, M)$ into $H^n(L, M)$. For any G-module U, let $U^G=\{u \mid \gamma\cdot u=0 \text{ for all } \gamma \text{ in } G\}$. A special case of a theorem of Hochschild-Serre [1, p. 603, Theorem 13][1] is

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

1.1. THEOREM. *If G/L is semi-simple then ρ_n is an isomorphism of $H^n(G, M)$ onto $H^n(L, M)^G$ for $n=0, 1, 2$.*

Hochschild-Serre's proof of their Theorem 13 depends on the theory of spectral sequences, and it seems desirable to sketch a direct proof of 1.1 here. Accordingly let $G=K+L$ be a Levi decomposition of G.

(1.2) *$H^n(L, M)^G = H^n(L, M)^K$ for all $n \geqq 0$.*

(1.3) *Every element of $H^n(L, M)^K$ has a representative cocycle in $C^n(L, M)^K$.*

(1.4)[2] *If $df \in C^n(L, M)^K$ for f in $C^{n-1}(L, M)$ then $\exists g$ in $C^{n-1}(L, M)^K \ni df = dg$.*

(1.5) *Let f be in $Z^n(G, M)$ where $n=1$ or 2 then there exists f^* in $Z^n(G, M) \ni f^*$ is cohomologous to f and*

(a) *$f^*(k)=0$ for all k in K (if $n=1$),*

(b) *$f^*(\gamma_1, \gamma_2)=0$ if γ_1 or γ_2 is in K (if $n=2$).*

PROOF. (a) If $f \in Z^1(G, M)$ then $f|_K \in Z^1(K, M)$ whence $\exists m$ in $C^0(K, M) = M \ni f(k) = k \cdot m$ for all k in K. Put

$$f^*(\gamma) = (f - dm)(\gamma) \text{ for all } \gamma \text{ in } G.$$

(b) If $f \in Z^2(G, M)$, define a Lie algebra extension (E, π) of M by G as follows: $E=(G, M)$ (direct product) as a vector space with $[(\gamma_1, m_1), (\gamma_2, m_2)] = ([\gamma_1, \gamma_2], \gamma_1 \cdot m_2 - \gamma_2 \cdot m_1 + f(\gamma_1, \gamma_2))$ for all γ_1, γ_2 in in G, m_1, m_2 in M; π the projection of E onto G.

Now $G=K+L$ with K semi-simple. $\pi^{-1}(L)$ is an ideal of E containing M and $E/\pi^{-1}(L) \cong K$. Thus we can find a linear map $\sigma: G \to E \ni \pi\sigma(\gamma) = \gamma$ for all γ in G and $\sigma|_K$ is a Lie algebra isomorphism of K into E. Consider $\pi^{-1}(L)$ and M as $\sigma(K)$-modules in the regular representation. Then since $\sigma(K)$ is semi-simple, there exists a submodule H of $\pi^{-1}(L) \ni \pi^{-1}(L) = H + M$ and $H \cap M = (0)$. $\pi|_H$ is a vector space isomorphism of H onto L so we see that we can choose the linear map σ so that

(1) $\sigma(k_1, k_2] = [\sigma(k_1), \sigma(k_2)]$ for all k_1, k_2 in K.

(2) $E = \sigma(K) + \sigma(L) + M$ direct sum as $\sigma(K)$-modules. Now note that

$$[\sigma(k), \sigma(x)] - \sigma[k, x] \in \sigma(L) \to M = (0) \text{ for all } k \text{ in } K, x \text{ in } L.$$

The cocycle $f^* \ni f^*(\gamma_1, \gamma_2) = [\sigma(\gamma_1), \sigma(\gamma_2)] - \sigma[\gamma_1, \gamma_2]$ is cohomologous to f and has the desired properties.

PROOF OF 1.1. If $n=0$ this is trivial. If $n=1$ or 2 and $h \in H^n(G, M)$, choose a representative cocycle f of h as in (1.5). It is easy to see that $k \cdot f|_L = 0$ for all k in K where $f|_L$ is the restriction of f to L so that

[2] (1.2), (1.3) and (1.4) are proved directly in [1, p. 603].

$\rho_n(H^n(G, M)) \subseteq H^n(L, M)^G$. Using (1.4) one sees quickly that ρ_n is an isomorphism.

To see that ρ_n is onto, take $\bar{h}$ in $H^n(L, M)^G$ $(n=1, 2)$ and choose a representative f of $\bar{h}$ in $Z^n(L, M)^K$ by (1.2) and (1.3). Extend f to $\hat{f}$ in $Z^n(G, M)$ by

$$n = 1 \qquad \hat{f}(k + x) = f(x) \qquad \text{for all } x \text{ in } L,\ k \text{ in } K;$$

$$n = 2 \qquad \begin{cases} \hat{f}(k, \gamma) = 0 & \text{for all } \gamma \text{ in } G,\ k \text{ in } K, \\ \hat{f}(x_1, x_2) = f(x_1, x_2) & \text{for all } x_1, x_2 \text{ in } L \end{cases}$$

then $\rho_n(\hat{f} + dC^{n-1}(G, M)) = \bar{h}$.

1.6. Definition. Let G be a Lie algebra and V an ideal of G. If there exists a subalgebra U of G such that $G = U + V$ and $U \cap V = (0)$ we say that G *splits over* V.

As a corollary of 1.2 we have

1.7. *Let G be a Lie algebra over a ground field of characteristic* 0 *and let $L \geqq M$ be ideals of G such that G/L is semi-simple, M is abelian and L splits over M. Then G splits over M.*

It would be interesting to know if the hypothesis that M be abelian can be removed from 1.7.

2. **Module extensions.** Let P and Q be G-modules, L any ideal of G. (We drop the requirement that G/L be semi-simple and that the ground field have characteristic 0.) If $P \subseteq E$ and if the sequence $0 \to P \to {}^iE \to {}^\phi Q \to 0$ of L-modules is exact we say that the pair (E, ϕ) is an L-extension of P by Q. Given two L-extensions (E_1, ϕ_1) and (E_2, ϕ_2) of P by Q, they are called L-equivalent if there is an L-isomorphism τ of E_1 onto E_2 such that $\tau(p) = p$ for all p in P and such that $\phi_2\tau = \phi_1$. We denote by $\{(E, \phi)\}$ the class of extensions L-equivalent to (E, ϕ) and recall that these classes form a vector space, over the ground field of L, which we denote by $\mathrm{Ext}_L\,[P, Q]$.

Let M be the vector space of linear maps of Q into P made into a G (and thus also an L)-module by setting

$$(\gamma \cdot f)(q) = \gamma \cdot f(q) - f(\gamma \cdot q) \text{ for all } \gamma \text{ in } G, f \text{ in } M$$

and q in Q. It is known that $\mathrm{Ext}_L\,[P, Q] \cong H^1(L, M)$ as a vector space [**2**, p. 562] and we now define an operation of G on $\mathrm{Ext}_L\,[P, Q]$ which agrees with the operation of G on $H^1(L, M)$ in the sense that this vector space isomorphism becomes a G-isomorphism. Take γ in G, and (E, ϕ) an L-module extension of P by Q. As a vector space, put

$$E_\gamma^* = \{(e_1, e_2) \mid e_1, e_2 \text{ in } E \text{ and } \gamma \cdot \phi(e_1) = \phi(e_2)\}.$$

Make E_γ^* into an L-module such that

$$x\cdot(e_1, e_2) = (x\cdot e_1, x\cdot e_2 + [\gamma, x]\cdot e_1) \qquad \text{for } x \text{ in } L \text{ and}$$

(e_1, e_2) in E_γ^*. Define a map ϕ_γ^* of E_γ^* onto Q by $\phi_\gamma^*(e_1, e_2) = \phi(e_1)$. The kernel of ϕ_γ^* is (P, P) and the map w_γ: $(P, P) \to P$ such that

$$w_\gamma(p_1, p_2) = \gamma\cdot p_1 - p_2$$

is an L-homomorphism of (P, P) onto P. The kernel N_γ of w_γ consists of all pairs (p_1, p_2) for which $p_2 = \gamma\cdot p_1$. N_γ is an L-submodule of E_γ^* so that we may use w_γ to induce an L-identification of $(P, P)/N_\gamma$ with P. Finally we put

$$\gamma\cdot(E, \phi) = (E_\gamma^*/N_\gamma, \phi_\gamma^*)$$

[ϕ_γ^* is used again to denote the L-homomorphism of E_γ^*/N_γ induced by the original ϕ_γ^*.]

Suppose now that (E_1, ϕ_1) is an extension of P by Q which is L-equivalent to (E, ϕ) and let ψ be an L-isomorphism of E onto E_1. The map $\hat{\psi}$: $E_\gamma^* \to E_{1\gamma}^*$ such that

$$\hat{\psi}(e_1, e_2) = (\psi(e_1), \psi(e_2))$$

is an L-homomorphism which takes N_γ onto $N_{1\gamma}$ and induces an L-isomorphism of E_γ^*/N_γ onto $E_{1\gamma}^*/N_{1\gamma}$. Thus we may put

$$\gamma\{(E, \phi)\} = \{\gamma\cdot(E, \phi)\}.$$

2.1. Proposition. *If h is the element of $H^1(L, M)$ determined by (E, ϕ) then the element of $H^1(L, M)$ determined by $\gamma\cdot(E, \phi)$ is $\gamma\cdot h$ [M = linear maps of Q into P].*

Proof. Let ψ be a linear map of Q into E such that $\phi\psi(q) = q$ for all q in Q. Then we know that the f in $C^1(L, M)$ such that

$$f(x)\{q\} = x\cdot\psi(q) - \psi(x\cdot q) \qquad \text{for all } x \text{ in } L$$

is an element of $Z^1(L, M)$ which represents h. Define a linear map ψ^* of Q into E_γ^*/N_γ by

$$\psi^*(q) = (\psi(q), \psi(\gamma\cdot q)) + N_\gamma.$$

Then $\phi_\gamma^*\psi^*(q) = q$ for all q in Q and

$$x\cdot\psi^*(q) - \psi^*(x\cdot q) = (\gamma\cdot f)(x)\{q\}$$

for all x in L.

2.2. Corollary. *The operation of G on $\mathrm{Ext}_L[P, Q]$ induces the structure of a G-module on $\mathrm{Ext}_L[P, Q]$ and the vector space isomorphism*

of $\mathrm{Ext}_L\,[P, Q]$ *onto* $H^1(L, M)$ *becomes a G-isomorphism.*

Now define a map $\rho\colon \mathrm{Ext}_G\,[P, Q]\to\mathrm{Ext}_L\,[P, Q]$ as follows. If (E, ϕ) is a G-extension of P by Q, define an L-extension $(\bar{E}, \bar{\phi})$ of P by Q by restricting the operators on E to elements of L. As an L-module, $\bar{E}\equiv E$ and as an L-homomorphism $\bar{\phi}\equiv\phi$. It is evident that if h is the element of $H^1(G, M)$ determined by $\{(E, \phi)\}$ then $\rho_1(h)$ is the element of $H'(L, M)$ determined by $\rho(\{(E, \phi)\})$ where ρ_1 is the restriction homomorphism. Thus we have:

2.3. Proposition. $\rho(\mathrm{Ext}_G\,[P, Q])\subseteq\mathrm{Ext}_L\,[P, Q]^G$.

Now we have the diagram:

$$\begin{array}{ccc} \mathrm{Ext}_G\,[P, Q] & \xrightarrow{\rho} & \mathrm{Ext}_L\,[P, Q]^G \\ \updownarrow & & \updownarrow \\ H(G, M) & \xrightarrow{\rho_1} & H^1(L, M)^G \end{array}$$

where ρ and ρ_1 are homomorphisms into. Further this diagram is commutative.

It is easy to find examples showing that ρ_1 (thus also ρ) need not be an isomorphism in general. Moreover, the following example (due to Hochschild) shows that ρ_1 need not even be onto in general: G is 3 dimensional, with basis x, y, z such that $[x, y]=z$ and z is central. Put $L=(z)$, M a trivial 1 dimensional G-module. Then every 1-cocycle for G in M must map L into (0); but the map f, where $f(az)=a$, represents a nonzero element of $H^1(L, M)^G$.

In fact it is an easy consequence of 1.1 and Hochschild-Serre's Theorem 6 [**1**, p. 596] that

2.4. Proposition. *If the ground field of G has characteristic* 0 *then* $\rho_1\colon H^1(G, M)\to H(L, M)^G$ *is an isomorphism for all G-modules M if and only if G/L is semi-simple.*

3. **Lie algebra extensions.** Let L be an ideal of G and M a G-module which we consider as an abelian Lie algebra. An extension of M by L is a pair (U, π) such that M is an ideal of U and such that the sequence $0\to M\to{}^iU\to{}^\pi L\to 0$ is exact. Two such extensions (U_1, π_1) and (U_2, π_2) are called equivalent if there is an isomorphism $\sigma\colon U_1\to U_2$ such that $\sigma(m)=m$ for all m in M and $\pi_1=\pi_2\sigma$. Using a procedure analogous to the Baer product, one can define directly the structure of a vector space on the classes of extensions $\{(U, \pi)\}$. We denote this vector space by $\mathrm{Ext}(M, L)$. Further it is known, [**2**, p. 566], that this vector space is isomorphic to $H^2(L, M)$.

We define an operation of G on $\mathrm{Ext}(L, M)$ as follows: take γ in

G and (U, π) an extension of M by L. Put

$$U_\gamma = \{\langle u_1, u_2\rangle \mid u_1, u_2 \text{ in } U, [\gamma, \pi(u_1)] = \pi(u_2)\}$$

as a vector space and make U_γ into a Lie algebra by setting

$$[\langle u_1, u_2\rangle, \langle u_1', u_2'\rangle] = \langle [u_1, u_2'], [u_1, u_2'] + [u_2, u_1']\rangle$$

for all $\langle u_1, u_2\rangle$ and $\langle u_1', u_2'\rangle$ in U_γ. Define a homomorphism π_γ of U_γ onto L by $\pi_\gamma\langle u_1, u_2\rangle = \pi(u_1)$. The kernel of π_γ is $\langle M, M\rangle$. This situation makes $\langle M, M\rangle$ into an L-module with L operating on $\langle M, M\rangle$ via U_γ. The map ω_γ: $\langle M, M\rangle \to M$ such that

$$\omega_\gamma\langle m_1, m_2\rangle = \gamma \cdot m_1 - m_2$$

is an L-homomorphism of $\langle M, M\rangle$ onto M. The kernel M_γ of ω_γ consists of pairs $\langle m, \gamma \cdot m\rangle$ and M_γ is an ideal of U_γ. Thus π_γ induces a homomorphism (still denoted by π_γ) of U_γ/M_γ onto L with kernel $\langle M, M\rangle/M_\gamma$ which we identify with M using ω_γ. Now define

$$\gamma \cdot (U, \pi) = (U^\gamma/M_\gamma, \pi_\gamma).$$

If (U_1, π_1) is equivalent to (U, π) under the isomorphism ψ, the map ψ^* such that $\psi^* - \langle u_1, u_2\rangle = \langle \psi(u_1), \psi(u_2)\rangle$ induces an equivalence isomorphism of $(U_\gamma/M_\gamma, \pi_\gamma)$ with $(U_{1\gamma}/M_{1\gamma}, \pi_{1\gamma})$ so that we may put

$$\gamma \cdot \{(U, \pi)\} = \{\gamma \cdot (U, \pi)\}.$$

3.1. Proposition. *Let h be the element of $H^2(L, M)$ determined by (U, π). Then the element of $H^2(L, M)$ determined by $\gamma \cdot \{(U, \pi)\}$ is $\gamma \cdot h$.*

Proof. Let α be a linear map of L into U such that $\pi\alpha(x) = x$ for all x in L. The element f of $C^2(L, M)$ such that

$$f(x_1, x_2) = [\alpha(x_1), \alpha(x_2)] - \alpha([x_1, x_2])$$

is in $Z^2(L, M)$ and represents h. Define a linear map $\hat{\alpha}$: $L \to U_\gamma/M_\gamma$ by

$$\hat{\alpha}(x) = \langle \alpha(x), \alpha([\gamma, x])\rangle + M_\gamma.$$

Then

$$[\hat{\alpha}(x_1), \hat{\alpha}(x_2)] - \hat{\alpha}[x_1, x_2] = (\gamma \cdot f)(x_1, x_2) \qquad \text{for all } x_1, x_2 \text{ in } L.$$

3.2. Corollary. *The operation of G on* $\mathrm{Ext}(M, L)$ *induces the structure of a G-module on* $\mathrm{Ext}(M, L)$ *and the vector space isomorphism of* $\mathrm{Ext}(M, L)$ *onto $H^2(L, M)$ becomes a G-isomorphism.*

Next consider $\mathrm{Ext}(M, G)$. If $\{(U, \pi)\}$ is in $\mathrm{Ext}(M, G)$ put

$$\rho\{(U, \pi)\} = \{(\pi^{-1}(L), \bar{\pi})\}$$

in $\mathrm{Ext}(M, L)$ where $\bar{\pi}$ is the restriction of π to $\pi^{-1}(L)$. ρ is a homomorphism of $\mathrm{Ext}(M, G)$ into $\mathrm{Ext}[M, L]$ and it is evident that the cohomology class in $H^2(L, M)$ belonging to $\rho\{(U, \pi)\}$ is $\rho_2(h)$ where g is the cohomology class in $H^2(G, M)$ belonging to $\{(U, \pi)\}$ and ρ_2 is the restriction homomorphism.

3.3. Proposition. ρ Ext $[M, G] \subseteq$ Ext $[M, L]^G$.

Proof. $\overline{U} = \pi^{-1}(L)$ is an ideal in U. Let $\bar{\pi} = \pi | \bar{u}$ then $\gamma \cdot (\overline{U}, \bar{\pi})$ is equivalent to the split extension (L, M) of M by L. In fact, the map

$$\langle \bar{u}_1, \bar{u}_2 \rangle + M_\gamma \to (\bar{\pi}(\bar{u}_1), \bar{u}_2 - [\gamma^1, \bar{u}_1])$$
$$(\bar{u}_i \text{ in } \overline{U}, \gamma' \text{ in } \overline{U} \text{ such that } \pi(\gamma') = \gamma)$$

of $\overline{U}_\gamma / M_\gamma$ onto (L, M) is an equivalence isomorphism.

Again we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}\,(M, G) & \xrightarrow{\rho} & \mathrm{Ext}\,(M, L)^G \\ \updownarrow & & \updownarrow \\ H^2(G, M) & \xrightarrow{\rho_1} & H^2(L, M)^G \end{array}$$

where ρ and ρ_2 are homomorphisms into. If G/L is semi-simple and if the ground field has characteristic 0 we know that ρ_2 is an isomorphism onto by 1.1. Thus

3.5. Proposition. *If G/L is semi-simple and if the ground field of G has characteristic* 0, *ρ is an isomorphism of* Ext (M, G) *onto* Ext $(M, L)^G$.

4. Lie algebra kernels. As before, let G be a Lie algebra, L an ideal of G and M a G-module. The derivations of L form a Lie algebra (denoted by $D(L)$) and the inner derivations form an ideal of $D(L)$ denoted by $I(L)$. An L-kernel with nucleus M is a pair (K, ϕ) consisting of a Lie algebra K and a homomorphism ϕ of L into $D(K)/I(K)$, M is required to be the center of K and ϕ must induce the given L-module structure on M.

Hochschild [**3**, p. 699 et seq.], has defined a rule of composition of L-kernels and a partitioning of the set of L-kernels with nucleus M into similarity classes such that the resulting set of similarity classes of kernels, Kern $[L, M]$, becomes a vector space over the same ground field as L. It is not difficult to show that each element of Kern $[L, M]$ determines a unique element of $H^3(L, M)$ and that this correspondence is in fact a vector space isomorphism of Kern $[L, M]$ into $H^3(L, M)$. This isomorphism is not onto in general and again we refer to [**3**] and [**4**] for the details.

Here we shall define directly an operation of G on Kern $[L, M]$ which is "correct" in the sense that if the similarity class $\{[K, \phi)\}$ determines h in $H^3(L, M)$ then $\gamma \cdot \{(K, \phi)\}$ determines $\gamma \cdot h$. An immediate consequence of this (at least if the ground field has characteristic 0) is that the effaceable elements of $H^3(L, M)$ constitute a G-submodule of $H^3(L, M)$. See [**4**, Theorem 5, p. 777].

In order to do this, let (K, ϕ) be an L-kernel with nucleus M where L is an ideal of G and take γ in G. Form the Lie algebra K_γ of pairs $\langle k_1, k_2\rangle$ with k_1, k_2 in K and put

$$[\langle k_1, k_2\rangle, \langle k_1', k_2'\rangle] = \langle [k_1, k_1'], [k_1, k_2'] + [k_2, k_1']\rangle \cdot$$

K_γ has center $\langle M, M\rangle$ and we define a homomorphism t_γ: $\langle M, M\rangle \to M$ such that $t_\gamma\langle m_1, m_2\rangle = \gamma \cdot m_1 + m_2$. The kernel T_γ of t_γ consists of pairs $\langle m, -\gamma \cdot m\rangle$ with m in M.

Now form K_γ/T_γ. The center of K_γ/T_γ is $\langle M, M\rangle/T_\gamma$; for if $[\langle a_1, a_2\rangle, \langle k_1, k_2\rangle]$ is in T_γ for all $\langle k_1, k_2\rangle$ in K_γ then $[a_1, k_1]$ is in M for all k_1 in K and $[a_1, k_2] + [a_2, k_1] = -\gamma \cdot [a_1, k_1]$. In particular, for pairs of the form $\langle m, k\rangle$ with m in M, we get

$$[a_1, k] + [a_2, m] = -\gamma \cdot [a_1, m] = 0$$

whence $[a_1, k] = 0$ for all k in K so that a_1 is in M. Thus since $[a_1, k_2] + [a_2, k_1] = -\gamma \cdot [a_1, k_1]$ for all $\langle k_1, k_2\rangle$, we see that $[a_2, k_1] = 0$ for all k_1 in K i.e. a_2 is in M.

We define a homomorphism ϕ_γ of L into $D(K_\gamma/T_\gamma)/I(K_\gamma/T_\gamma)$ as follows: let ρ be any linear map of L into $D(K)$ which is compatible with ϕ (i.e. the class mod $I(K)$ of $\rho(x)$ is $\phi(x)$ for all x in L.) Set

$$\rho^*(x)\{\langle k_1, k_2\rangle + T_\gamma\} = \langle \rho(x)\{k_1\}, \rho(x)\{k_2\} - \rho([\gamma, x])\{k_1\}\rangle + T_\gamma.$$

It is easy to check that ρ^* is a linear map of L into $D(K_\gamma/T_\gamma)$ and that the class mod $I(K_\gamma/T_\gamma)$ of $\rho^*(x)$ does not depend on the choice of ρ compatible with ϕ. Now we note that if

$$[\rho(x_1), \rho(x_2)] - \rho([x_1, x_2]) = D_{\tau(x_1, x_2)} \qquad \text{in } I(K) \text{ for } x_1, x_2 \text{ in } L,$$

then

$$[\rho^*(x_1), \rho^*(x_2)] - \rho^*([x_1, x_2]) = D_{\tau^*(x_1, x_2)} \qquad \text{in } I(K_\gamma/T_\gamma)$$

where $\tau^*(x_1, x_2) = \langle \tau(x_1, x_2), -\tau([\gamma, x_1], x_2) - \tau(x_1, [\gamma, x_2])\rangle + T_\gamma$. Now put $\phi_\gamma(x) = \phi^*(x) + I(K_\gamma/T_\gamma)$. One sees immediately that ϕ_γ induces the given L-module structure on the center $\langle M, M\rangle/T_\gamma$ and that t_γ induces an L-isomorphism of $\langle M, M\rangle/T_\gamma$ onto M so that the pair $(K_\gamma/T_\gamma, \phi_\gamma)$ is an L-kernel with nucleus M which we denote by $\gamma \cdot (K, \phi)$.

The function $\tau: L\times L\to K$ is called a hindrance of (K, ϕ) and its (formal) coboundary $\delta\tau$ is an element of $Z^3(L, M)$ called a primary deviation. The cohomology class of $\delta\tau$ (called the obstruction of (K, ϕ)) is independent of the choice of ρ compatible with ϕ and of the choice of τ such that $[\rho(x_1), \rho(x_2)]-\rho([x_1, x_2])=D_{\tau(x_1,x_2)}$.

In this terminology, $\tau^*(x_1, x_2)$ is a hindrance of $\gamma\cdot(K, \phi)$ and a straightforward computation shows that

$$\delta\tau^* = \gamma\cdot\delta\tau.$$

Since two L-kernels have the same obstruction if and only if they are similar, we have

4.1. Proposition. *The map* $\{(K, \phi)\}\to\{\gamma\cdot(K, \phi)\}$ *(where braces indicate similarity class) induces the structure of a G-module on Kern* $[L, M]$ *and further, if h is the obstruction of* $\{(K, \phi)\}$, *then* $\gamma\cdot h$ *is the obstruction of* $\{\gamma\cdot(K, \phi)\}$.

As before we define a map ρ of Kern $[G, M]$ into Kern $[L, M]$ by setting $\rho\{(\hat{K}, \hat{\phi})\} = \{(\hat{K}, \hat{\phi}|_L)\}$ where $\hat{\phi}|_L$ is the restriction of $\hat{\phi}$ to L. It is evident that if h is the obstruction of $\{(\hat{K}, \hat{\phi})\}$ then $\rho_3(h)$ is the obstruction of $\{(\hat{K}, \hat{\phi}|_L)$ where ρ_3 is the restriction map of $H^3(G, M)$ into $H^3(L, M)$. Again we have a commutative diagram:

$$\begin{array}{ccc} H^3(G, M) & \to & H^3(L, M)^G \\ \downarrow & & \downarrow \\ \text{Kern } [G, M] & \to & \text{Kern } [L, M]^G \end{array}$$

Appendix

For details concerning the following discussion the reader is referred to [**5**].

A. Group enlargements.

Definition. Let G be an arbitrary group (written multiplicatively) and P, Q groups on which G operates from the left. In this section, P is assumed commutative but Q need not be. A G-enlargement of P by Q is a pair (E, ϕ) such that

(i) G operates on E from the left,

(ii) P is a subgroup of E and a direct summand of E (as a group),

(iii) ϕ is a G-homomorphism of E onto Q with kernel P.

Two G-enlargements (E_1, ϕ_1), (E_2, ϕ_2) are called *equivalent* if there exists a G-isomorphism $\tau: E_1\to E_2$ such that $\tau(p)=p$ for all p in P and $\phi_2\tau=\phi_1$.

Eilenberg gives a definition of multiplication of equivalence classes

of G-enlargements so that these classes form an abelian group denoted by Enl (G, Q, P). If θ is an element of Hom (Q, P) and if one puts $(\sigma\cdot\theta)\{q\}=\sigma\cdot\theta(\sigma^{-1}q)$ for all σ in G and q in Q then Hom (Q, P) becomes a G-module and Eilenberg then showed that ENL (G, Q, P) $\cong H^1(G, \text{Hom}\,(Q, P))$.

Now suppose that G is a normal subgroup of a group Γ and that Γ also operates on P and Q. Let (E, ϕ) denote a G-enlargement of P by Q. Take γ in Γ and let E_γ be the subgroup of $E\times E$ consisting of pairs (e_1, e_2) such that $\phi(e_1)=\gamma\cdot\phi(e_2)$. Make E_γ into a G-module such that $\sigma\cdot(e_1, e_2)=(\sigma\cdot e_1, \sigma^\gamma e_2)$ where $\sigma^\gamma=\gamma^{-1}\sigma\gamma$ for all σ in G. Let ϕ_γ^* denote the G-homomorphism $(e_1, e_2)\to\phi(e_1)$ of E_γ onto Q. The kernel of ϕ_γ^* is (P, P) and the map α_γ: $(P, P)\to P$ such that $\alpha_\gamma(p_1, p_2)=\gamma\cdot p_2$ is a G-homomorphism of (P, P) whose kernel ker (α_γ) consists of pairs $(p, 1)$ where 1 is the identity of P. ϕ_γ^* induces a G-homomorphism ϕ_γ of $E_\gamma/\text{ker}\,(\alpha_\gamma)$ onto Q so that the pair $(E_\gamma/\text{ker}\,(\alpha_\gamma), \phi_\gamma)$ is a G-enlargement of P by Q which we denote by $\gamma\cdot(E, \phi)$. It is not difficult to show that if (E, ϕ) is equivalent to (E', ϕ') then $\gamma\cdot(E, \phi)$ is equivalent to $\gamma\cdot(E', \phi')$ and that this operation induces the structure of a Γ-module on Enl (G, Q, P). Further one can check that this structure is compatible with the Γ-module structure on $H^1(G,$ Hom $(Q, P))$ so that the natural isomorphism of Enl (G, Q, P) onto $H^1(G,$ Hom $(Q, P))$ becomes a Γ-isomorphism.

B. **Group extensions.** If M is a subgroup of U and if the sequence $0\to M\to U\to^\pi G\to 0$ is exact then we say that the pair (U, π) is an extension of G by M. If M is abelian, this situation makes M into a G-module. Two such extensions (U_1, π_1) and (U_2, π_2) of G by M are called equivalent if there exists an isomorphism τ: $U_1\to U_2$ such that $\tau(m)=m$ for all m in M and such that $\pi_1=\pi_2\tau$. The equivalence classes of such extensions form an abelian group $A(G, M)$ which is isomorphic with $H^2(G, M)$ see [**5**].

Now suppose that G is a normal subgroup of a group Γ and that M is also a Γ-module. Let (U, π) be an extension of G by M and take γ in Γ. Form $U_\gamma=\{\langle u_1, u_2\rangle \mid u_i \text{ in } U,\ \pi(u_1)=\pi(u_2)^\gamma\}$ with $\langle u_1, u_2\rangle\langle u_1', u_2'\rangle=\langle u_1u_1', u_2u_2'\rangle$. Set $\pi_\gamma\langle u_1, u_2\rangle=\pi(u_1)$ and note that the kernel of π_γ is $\langle M, M\rangle$. Define a homomorphism ω_γ: $\langle M, M\rangle\to M$ by $\omega_\gamma\langle m_1, m_2\rangle=\gamma\cdot m_2$ for all $\langle m_1, m_2\rangle$ in M. ω_γ is a G-homomorphism of $\langle M, M\rangle$ onto M and the kernel of $\omega_\gamma\equiv\langle M, 0\rangle$ is a normal subgroup of $U_\gamma\cdot\pi_\gamma$ induces a homomorphism (still denoted by π_γ) of $U_\gamma/\langle M, 0\rangle$ onto M and we put

$$\gamma\cdot(U, \pi) = (U_\gamma/\langle M, 0\rangle, \pi_\gamma).$$

One can check that if (U, π) is equivalent to (U_1, π_1) then $\gamma\cdot(U, \pi)$

is equivalent to $\gamma\cdot(U_1, \pi_1)$ and that this operation induces the structure of a Γ-module on $A(G, M)$. Further, if h is the element of $H^2(G, M)$ belonging to (U, π) then $\gamma\cdot h$ belongs to $\gamma\cdot(U, \pi)$ so that the isomorphism of $A(G, M)$ onto $H^2(G, M)$ becomes a Γ-isomorphism.

C. **Group kernels.** A G-kernel is a pair (K, θ) where θ is a homomorphism: $G\rightarrow A(K)/I(K)$ (the outer automorphisms of K). If M is the center of K, then θ makes M into a G-module called the nucleus of (K, θ). Eilenberg-MacLane [**5**, p. 12] have given a rule of composition of G-kernels with fixed nucleus M and a partitioning of the set of G-kernels with nucleus M into similarity classes such that the resulting set of similarity classes $\mathcal{K}(G, M)$ becomes an abelian group. They then show that $\mathcal{K}(G, M)$ is isomorphic to $H^3(G, M)$.

Now suppose that G is a normal subgroup of a group Γ and that M is also a Γ-module. Let (K, θ) be a G-kernel with nucleus M and take γ in Γ. As a group, put $K_\gamma = K$ and put $\theta_\gamma(\sigma) = \theta(\gamma^{-1}\sigma\gamma)$ for all σ in G. The center M_γ of K_γ is still M (as a group) but now the operation of G on M_γ is given by

$$\sigma * m = \gamma^{-1}\sigma\gamma\cdot m \qquad \text{for all } \sigma \text{ in } G,\ m \text{ in } M.$$

The map $m\rightarrow\gamma\cdot m$ is a G-isomorphism of M_γ onto M which we use to identify M_γ with M as a G-module. Then the pair $(K_\gamma, \theta_\gamma)$ is a G-kernel with nucleus M which we denote by $\gamma\cdot(K, \theta)$. One can check that this operation induces the structure of a Γ-module on $\mathcal{K}(G, M)$ and sees that if h is the element of $H^3(G, M)$ belonging to (K, θ) then $\gamma\cdot h$ belongs to $\gamma\cdot(K, \theta)$ so that the isomorphism of $\mathcal{K}(G, M)$ with $H^3(G, M)$ becomes a Γ-isomorphism.

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