NONCOUNTABLE NORMALLY LOCALLY
FINITE DIVISION ALGEBRAS

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A (commutative) field $F$ is regular (see [1] of the bibliography) if it
is not finite, and if in addition it is true that the direct (= Kronecker
= tensor) product of two normal (= central) division algebras, of
finite orders, over $F$ is not a division algebra unless their orders are
relatively prime; algebraic number fields and $p$-adic fields are ex-
amples of regular fields. A division algebra $A$ over a field $F$ is normally
locally finite if any finite subset of $A$ is contained in a normal (over $F$)
division sub-algebra of $A$ of finite order; in [1], such algebras were
called "of type 1." A subisomorphism of an algebra $A$ over $F$ is an
algebra-isomorphism of $A$ into $A$, and it is proper if it is not onto. If
$A$ is a normally locally finite division algebra over the regular field $F$,
without a finite basis over $F$, a characteristic sub-algebra of $A$ is any
normally locally finite division sub-algebra $D$ of $A$, with countably
infinite basis over $F$, having the property that any normally locally
finite division sub-algebra of $A$, with finite or countable basis over $F$,
is isomorphic to a sub-algebra of $D$. It was proved in [1] that any $A$
of the previous type has a characteristic sub-algebra, unique but for
isomorphisms; it was also proved that there exists a normally locally
finite division algebra over the regular field $F$, with infinite non-
countable basis, and with a given characteristic sub-algebra $D$, if
and only if $D$ admits proper subisomorphisms; [1] contains a rather
involved proof of the fact that any $D$ admits proper subisomorphisms
if $F$ is not countable, and thus establishes the existence of normally
locally finite division algebras, with infinite noncountable basis, over
any noncountable regular field; this seems to be the only known
example of such algebras. We shall present here a very simple proof
of the same result, and will, at the same time, dispense with the
condition of noncountability of $F$.

(1). Lemma. Let $A$, $B$, $C$ be normal division algebras, of finite orders
$>1$, over the (certainly infinite) field $F$, and suppose $A \times B \times C$ also
to be a division algebra; let $m$ be an element of $A \times B$ but not of $A$. Then
there exists a $d \in B \times C$, not zero, such that $d^{-1}md \in A \times B$.

In the previous statement, as in the rest of this paper, the identity

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elements of the direct factors of a direct product of algebras are assumed to be coincident.

Proof (being a modification, due to D. Zelinsky, of a proof by the author). Let $c$ be an element of $C$, but not of $F$, and let $b$ be an element of $B$ such that $mb \neq bm$; such $b$ exists because the commutator (= centralizer) of $B$ in $A \times B$ is $A$, and $m \in A$. Set $x = bc \neq 0$, so that also $1 + x \neq 0$; if the lemma is false, we have $xy = mx$, $(1 + x)z = m(1 + x)$ for suitable elements $y, z$ of $A \times B$; subtracting, we obtain $m = z + x(z - y)$. If $y = z$, then $m = z = y, xm = mx, bm = mb$, a contradiction; if $y \neq z$, then $bc = x = (m - z)(z - y)^{-1} \in A \times B$, also a contradiction, since $c \in F$, Q.E.D.

For the convenience of the reader, we repeat here a portion of the statement of (6) of [1]:

(2). Lemma. Let $A$ be a normally locally finite division algebra, with countably infinite basis, over the field $F$; a necessary and sufficient condition in order that $A$ admit a proper subisomorphism is that there exist a factorization

$$A = B_0 \times B_1 \times \cdots$$

of $A$ as a direct product of normal division algebras of finite orders $> 1$ over $F$, an $m \in B_0$, and a sequence $h_1, h_2, \cdots$ of elements of $A$, such that, after setting $A_i = B_0 \times B_1 \times \cdots \times B_i$, the following conditions be satisfied:

(a) $h_i \in A_i$;
(b) $h_{i+1} = h_i c_i$ for a $c_i \in B_i \times B_{i+1}$;
(c) there exists no $z_{i-1} \in A_{i-1}$ such that $h_i z_{i-1} = m z_i$.

We can now prove:

(3). Theorem. Let $A$ be as in (2); then $A$ admits a proper subisomorphism.

Proof. By (29) of [1], $A_0$ cannot be transformed into itself by every inner automorphism of $A_1$; hence there exist an $m \in A_0$, and an $h_1 \in A_1$, with $h_1 \neq 0$, such that $h_1^{-1} mh_1 \in A_0$. We shall now proceed to build the sequence $\{ h_i \}$ of (2) by induction: assume the $h_1, \cdots, h_i$ to have been found; by (1) (after replacing $A$ by $A_{i-1}$, $B$ by $B_i$, $C$ by $B_{i+1}$, $m$ by $h_i^{-1} mh_i$), there exists a $c_i \in B_i \times B_{i+1}$, not zero, such that $c_i^{-1} (h_i^{-1} mh_i) c_i \in A_i$. Then $h_{i+1} = h_i c_i$ satisfies the conditions of (2), Q.E.D.

1 This is the little theorem with a distinguished career, first proved in [2], which later came to be known as the Cartan-Brauer-Hua theorem (see for instance [3, Chapter VII, §13]); the proof given in [1] is the first elementary proof for the finite case.
(4). Corollary. Let $A$ be as in (2), and assume $F$ to be regular; then there exists a normally locally finite division algebra over $F$, with infinite noncountable basis, having $A$ as characteristic sub-algebra.

On the other hand, if $F$ is not regular, the concept of characteristic sub-algebra loses meaning; however, from (3), and from a slight modification of the construction used to prove the sufficiency of (3) of [1], we still obtain:

(5). Corollary. Let $F$ be a field such that there exists a normally locally finite division algebra $A$ with countably infinite basis over $F$; then there exists a normally locally finite division algebra over $F$, with infinite noncountable basis, having $A$ as a sub-algebra.

Remark. An examination of the proof of (3) of [1] discloses that all the normally locally finite division algebras, with infinite noncountable basis over $F$, whose existence has been established in this note, have a basis of cardinality $\aleph_1$; the existence of normally locally finite division algebras over $F$, with a basis of cardinality $>\aleph_1$, is still an open problem, at least when $F$ is regular.

Bibliography


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