ON THE COMPOSITION FUNCTIONS OF NILPOTENT LIE GROUPS

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This note contains a proof of the theorem:

If the composition functions of a real or complex Lie group are polynomials, then it is nilpotent.

The converse theorem is found among E. Cartan's works [2] and can be also shown through the Baker-Hausdorff formula in a rather straightforward fashion (see [1]).

Let $\mathfrak{G}$ be a real or complex Lie group with Lie algebra $\mathfrak{g}$. Choose for $\mathfrak{g}$ a base $X_1, \cdots, X_n$, dual to which there is a base $\omega_1, \cdots, \omega_n$ of the Maurer-Cartan forms. Denote by $\mathbf{x}=(x_1, \cdots, x_n)$ the corresponding canonical coordinate system and by $f_i(\mathbf{x}, \mathbf{y})$, $i=1, \cdots, n$, the composition functions with respect to the coordinate system. If $f_i(\mathbf{x}, \mathbf{y})$, $i=1, \cdots, n$, are polynomials, we are going to prove that $\mathfrak{G}$ is nilpotent.

For $X\in\mathfrak{g}$, define $R(X)$ to be the linear transformation (operating on the right side) of the vector space of $\mathfrak{g}$ such that, for $Y\in\mathfrak{g}$,

$$YR(X) = [Y, X] = YX - XY.$$

Allow the vector space of $\mathfrak{g}$ to be normed so that the norm $|R(X)|$ of $R(X)$ is defined.

It is known that, about the identity e of $\mathfrak{G}$, the Maurer-Cartan forms

$$\omega_i(\mathbf{x}, d\mathbf{x}) = \sum A_{ij}(\mathbf{x})dx_j, \qquad i = 1, \cdots, n,$$

are explicitly given through the formulas

$$\sum A_{jk}(\mathbf{x})X_j = X_k \frac{\exp R(X) - 1}{R(X)}, \qquad k = 1, \cdots, n,$$

where $X = \sum x_i X_i$.

We assert that, about e, the left invariant infinitesimal transformations $X_i = \sum H_{ij}(\mathbf{x})\partial/\partial x_j$, $i=1, \cdots, n$, can be explicitly given through the formulas

$$\sum H_{ij}(\mathbf{x})X_j = X_i \frac{R(X)}{\exp R(X) - 1}.$$

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In fact, if $\overline{X}_i=\sum H_{ij}(\mathbf{x})\partial/\partial x_j$, $i=1, \cdots, n$, where the H_{ij} are defined by the preceding equation, then we have about e,

$$\begin{aligned}\sum \omega_i(\overline{X}_j)X_i &= \sum_{i,k} A_{ik}(\mathbf{x})H_{jk}(\mathbf{x})X_i \\ &= \sum H_{jk}(\mathbf{x})X_k\frac{\exp R(X)-1}{R(X)} \\ &= X_j\frac{R(X)}{\exp R(X)-1}\,\frac{\exp R(X)-1}{R(X)}\,.\end{aligned}$$

Since the series respectively represented by $R(X)/(\exp R(X)-1)$ and $(\exp R(X)-1)/R(X)$ converge both absolutely, we obtain $\sum\omega_i(\overline{X}_j)X_i=X_j$, i.e. $\omega_i(\overline{X}_j)=\delta_{ij}$. Therefore each $\overline{X}_i$ coincides with X_i about e.

On the other hand, we observe that each $H_{ij}(\mathbf{x})$ is equal to $(\partial f_i(\mathbf{x}, y)/\partial y_j)_{y=0}$ which is a polynomial, say, of degree $\leqq N$. Let $R(X)/(\exp R(X)-1)$ be expanded in the series $\sum_{p=0}^{\infty} B_p[R(X)]^p$ and set $B_pX_i[R(X)]^p=\sum H_{ijp}(\mathbf{x})X_j$. Then each $H_{ijp}(\mathbf{x})$ is a homogeneous polynomial of degree p, and

$$H_{ij}(\mathbf{x}) = \delta_{ij}+\sum_{p=1}^{\infty} H_{ijp}(\mathbf{x}).$$

We conclude that $B_pX_i[R(X)]^p=0$, $i=1, \cdots, n$, and therefore $B_pY[R(X)]^p=0$ for any $Y\in\mathfrak{g}$ when $p>N$. Since there exists $q>N$ with $B_q\neq 0$, this yields $Y[R(X)]^q=0$ for any $X, Y\in\mathfrak{g}$. According to a theorem due to Engel (see [**3**]), $\mathfrak{g}$ is nilpotent. Hence $\mathfrak{G}$ is nilpotent.

Bibliography

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3. H. Zassenhaus, *Über Liesche Ringe mit Primzahlcharakteristik*, Abh. Math. Sem. Hanischen Univ. vol. 13 (1939) pp. 1–100.

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