NOTE ON A PAPER OF DIEUDONNÉ

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In a recent paper [1], Dieudonné has proved the following theorem.

Let \( a_0, a_1, a_2, \cdots \) be an infinite sequence of rational numbers and consider the (formal) power series

\[
\exp \left( \sum_{m=0}^{\infty} a_m x^m \right) = \sum_{n=0}^{\infty} c_n x^n,
\]

where \( p \) is a fixed prime. Then a necessary and sufficient condition that all the coefficients \( c_n \) be \( p \)-adic integers is that for each \( i \geq 0 \) we have

\[
a_i = \frac{a_{i-1}}{p} + b_i \quad (a_{-1} = 0),
\]

where each \( b_i \) is a \( p \)-adic integer.

In this note we consider the following problem. Let \( a_1, a_2, a_3, \cdots \) be a sequence of rational numbers and define the (formal) power series

\[
(1) \quad \exp \left( \sum_{m=1}^{\infty} a_m x^m \right) = \sum_{n=0}^{\infty} c_n x^n.
\]

Then we prove the following

**Theorem 1.** A necessary and sufficient condition that all the coefficients \( c_n \) be rational integers is that for all \( k \geq 1 \) we have

\[
(2) \quad \sum_{rs=k} r a_r \mu(s) \equiv 0 \pmod{k},
\]

where \( \mu(s) \) is the Möbius function.

To prove the theorem we employ a device used by Schur [2] and credited to Jänichen. Let

\[
g(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

Then we can recursively determine \( b_1, b_2, \cdots \) so that

\[
g(x) = \prod_{m=1}^{\infty} (1 - x^m)^{b_m}.
\]

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Moreover the $b_m$ are all integral if and only if the $c_n$ are all integral. Now from (1) and (3) it follows that

$$\sum_{k=1}^{\infty} a_k x^k = \log g(x) = - \sum_{m=1}^{\infty} b_m \sum_{r=1}^{\infty} \frac{x^{mr}}{r};$$

consequently

$$k a_k = - \sum_{m | k} m b_m.$$

But (4) is equivalent to

$$- b_k = \frac{1}{k} \sum_{rs=k} r a_r \mu(s).$$

As already noted, the $c_n$ are all integral if and only if the $b_k$ are all integral, that is, if and only if the right member of (5) is integral for all $k$. But this is the same as the condition (2). This evidently proves the theorem.

The theorem can be stated in a slightly more general way. Let $P$ denote an arbitrary set of rational primes and let $D$ denote the set of rational numbers whose denominators contain only the primes of $P$. Then we may state

**Theorem 2.** A necessary and sufficient condition that all the coefficients $c_n \in D$ is that for all $k \geq 1$

$$\left\{ \frac{1}{k} \sum_{rs=k} r a_r \mu(s) \right\} \subseteq D.$$

Dieudonné's theorem is obtained when $P$ consists of all the primes except $p$ and $a_m = 0$ except for $m = p^r$.

**References**
