D-REGULARITY

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We shall call an element x of a ring A, right D-regular if there exists an element y in A such that $x=xy$. This property of x belonging to xA has been studied before [2; 6].[1] With techniques available from [5] it is not difficult to show the existence of a maximal right D-regular two-sided ideal M_R and a left analogue M_L. These are in general not equal. They are connected with the Jacobson radical J and the subradicals P_R and P_L, [6], in the following way:

$$P_R = J \cap M_R; \qquad P_L = J \cap M_L.$$

The present note goes on to consider the cases $M_R=0$ and $M_R=A$, for various degrees of chain assumptions. In the commutative case $M_R=M_L=M$, the maximal D-regular ideal.

1. **Preliminaries.** Following Brown and McCoy [5], to each element a of a ring A we associate the right ideal $F(a)=aA$. The element a is said to be right D-regular (r.D.r.) if a belongs to $F(a)$. A right ideal is said to be r.D.r. if every element in it is r.D.r. It is easy to see that $F(a+b)\leqq F(a)+(b)_r\leqq F(a)+(b)$, and that $F(a+b)\leqq F(a)$ if b is in $F(a)$. Then by Theorems 1 and 2 of [5] we can conclude that $M_R = \{x: (x) \text{ is r.D.r.}\}$, is a two-sided ideal which contains every r.D.r. two-sided ideal; and that $M_R(A-M_R)=0$. In other words, A is an (F, Ω, Ω') group and by Theorem 6 of [5] we have:

$$M_R = \bigcap_i M_i', \tag{1}$$

where M_i is a large modular right ideal, i.e. there exists an element x_i not in M_i such that $x_iA\leqq M_i$ and such that every right ideal which properly contains M_i, also contains x_i. The set M_i' is the largest two-sided ideal contained in M_i.

Though this development is both elegant and general it does not seem to yield the fact that M_R contains all the r.D.r. right ideals. In particular, if x is in xA and if for every y of A, xy is in xyA, it is not clear that x must be in M_R. To obtain this fact one must return to the original Jacobson techniques and develop M_R from a one sided point of view. Using a technique of [6] we obtain:

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

LEMMA 1. *If x is r.D.r. and if a belongs to an r.D.r. right ideal, namely aA, then $a+x$ is r.D.r.*

PROOF. We have $xx'=x$, $aa'=a$. Define $u=a-ax'=aa'-ax'=a(a'-x')$. Then u belongs to aA and therefore there exists an element u' such that $uu'=u$. Define $v=u'+x'-x'u'$. Then $xv=xu'+xx'-xx'u'=xu'+x-xu'=x$; and $av=au'+ax'-ax'u'=(a-ax')u'+ax'=a-ax'+ax'=a$. Therefore $(a+x)v=a+x$. Q.E.D.

COROLLARY. *The sum or two r.D.r. right ideals is an r.D.r. right ideal.*

We define M_R^* to be the union of all the r.D.r. right ideals. Then M_R^* is itself an r.D.r. right ideal. We now show that it is a two-sided ideal:

LEMMA 2. *M_R^* is a two-sided ideal of A.*

PROOF. For a in A and x in M_R^*, xa is in M_R^*. Since $xx'=x$ and for every y of A, there is an element y' such that $xy=xyy'$; $axx'=ax$ and $axyy'=axy$. Therefore ax is in M_R^*. Q.E.D.

It is now easy to see that $M_R^*=M_R$ and we have

THEOREM 1. *If x is in xA and for every y of A, xy is in xyA, then x is in M_R.*

The proof of the next theorem is immediate from results in [6].

THEOREM 2. *If J is the Jacobson radical of A, then $P_R=J\cap M_R=J(M_R)$; and $P_L=J\cap M_L=J(M_L)$.*

Thus in the commutative case, the subradical is the radical of the max.D.r. ideal.

We shall now obtain some simple properties of M_R. First we observe that M_R and M_L may not be equal. The following example is due to Hopkins [8]. Let A be the set of all $me+nu$, where $e^2=e$, $u^2=0$, $eu=u$, $ue=0$, and where m, n are in a field F. Then $M_L=A$ whereas $M_R=0$. The proof of the following lemma is immediate.

LEMMA 3. *If B is an ideal of A, then $M_R(B)<B\cap M_R$.*

The point is that they may not be equal. Let $B=J$, and let A have a right unity element. Then $M_R(B)=0$, whereas $B\cap M_R=J\cap A=J$.

We observe also that $M_R=M_RA=M_RA^n$ for every n; $M_L=A^mM_L$, for every m.

LEMMA 4. *$M_R(M_R)\leqq M_R^n$ for every n; $M_R(M_R)=M_R(M_R^n)$ for every n.*

PROOF. If x is in $M_R(M_R)$, then in particular $x=xy$, with y in M_R.

Then $x=xy^{n-1}$, which is in M_R^n. The second half is also immediate, $x=xy^n$, with y^n in M_R^n.

Thus we have $M_R > M_R^2 > \cdots > M_R^n > \cdots > M_R(M_R) = M_R(M_R^2) = \cdots = M_R(M_R^n) = \cdots$.

This chain can in fact descend.

EXAMPLE 1. Let A be the set of all finite sums $\sum_1^n \alpha_i x^i + \sum_1^m \beta_j y^j$, where α_i and β_j are in a field F and where x and y are indeterminates such that $x^i y^j = y^j x^i = x^i$ for every i and j. Then $M = (x)$, $M^2 = (x^2), \cdots, M^n = (x^n)$, $M(M) = \cdots = M(M^n) = \cdots = 0$.

By methods almost the same as in [4], one can prove

THEOREM 3. *If A_n is the complete matric ring of order n over A then*

$$M_R(A_n) = (M_R(A))_n; \qquad M_L(A_n) = (M_L(A))_n.$$

This leads to

LEMMA 5. *If $a_1, \cdots, a_n$ is any finite set of elements in M_R, $a_i = a_i a_i'$, then there exists an element b in A but not necessarily in M_R such that $a_i = a_i b$ for all the a_i.*

PROOF. Since the a_i are in M_R, the $n \times n$ matrix c, with the a_i in the first column and zeros elsewhere, is in $M_R(A_n)$, by Theorem 3. Then, in particular, there exists a matrix $d = (b_{ij})$ such that $cd = c$. However

$$cd = \begin{pmatrix} a_1 b_{11} & \cdots \\ a_2 b_{11} & \cdots \\ \cdot & \\ \cdot & \\ \cdot & \\ a_n b_{11} & \cdots \end{pmatrix}$$

and therefore $b = b_{11}$.

An inductive proof can also be given. For $n=2$, Lemma 5 is a consequence of the proof of Lemma 1. We assume the result for any set of $n-1$ elements of M_R. Given $a_1, \cdots, a_n$, with $a_i = a_i a_i'$, consider the set of $n-1$ elements $a_i - a_i a_n'$, for $i = 1, \cdots, n-1$. By induction there exists an element g such that $(a_i - a_i a_n')g = a_i - a_i a_n'$. Define $b = a_n' + g - a_n' g$. Then $a_n b = a_n + a_n g - a_n g = a_n$; whereas $a_i b = a_i a_n' + (a_i - a_i a_n')g = a_i a_n' + a_i - a_i a_n' = a_i$, for $i = 1, \cdots, n-1$. Q.E.D.

COROLLARY. *If M_R is finitely generated as a left A-module, then there exists an element e in A such that $M_R = M_R e$ pointwise. The element e is not necessarily in M_R, or an idempotent or unique.*

It is clear that if A has a right unity element then $A = M_R$. If

$A=J$, then both M_R and M_L are zero, since x in xJ or Jx implies $x=0$, [3]. Thus we might expect that if $M_R=A$ then A is well behaved whereas if $M_R=M_L=0$ then A is radical-like. This is certainly so when A is commutative with DCC. In that case A can be expressed as $A=eA+N_0$ where e is an idempotent, $eN_0=0$ and N_0 is nilpotent. Since a central idempotent is always in both M_R and M_L, here e is in M and thus $eA \leqq M$. On the other hand if x is in M, $x=ex_1+n_1$, then in particular there exists an element $ey+n_2$ such that $(ex_1+n_1)\cdot(ey+n_2)=ex_1+n_1$. Then $ex_1y+n_1n_2=ex_1+n_1$ and thus $n_1n_2=n_1$, $n_1=0$. Therefore $M \leqq eA$, $M=eA$. Also N_0 is simply M', the set of annihilators of M. We have

THEOREM 4. *If A is a commutative ring with DCC then $A=M+M'$, where M, the max.D.r. ideal, has a unity element and M' is nilpotent.*

This corresponds to the result in [4] which states that every ring A with DCC (though not necessarily commutative) can be expressed as $\overline{M}+\overline{M}'$ where $\overline{M}$, the max. regular ideal, is semi-simple, and where $\overline{M}'$ is bound to its radical. Here $M>\overline{M}$ and $M'<\overline{M}'$. Thus we know more about M', namely that it is nilpotent, and less about M, since it is not necessarily semi-simple.

COROLLARY 1. *If A is commutative with DCC then $M=A$ if and only if A has a unity element; $M=0$ if and only if A is nilpotent.*

Using the fact that $M(A-M)=0$ we then have

COROLLARY 2. *If A is commutative with DCC for $A-M$ or in particular for A, then $A-M$ is nilpotent.*

Theorem 4 and its corollaries remain true if the condition of commutativity is relaxed to the restriction that all idempotents lie in the center. Without DCC however, Theorem 4 is false, for in Ex. 1, M is the set of all $\sum_1^n \alpha_i x^i$, whereas M' is the set of all $\sum_1^m \beta_j y^j$ with $\sum_1^m \beta_j=0$. Thus M and M' do not fill out all of A.

2. The cases M_R, M_L, M_R and M_L equal to zero.

THEOREM 5. *If A is a ring with DCC on right ideals, then A is nilpotent if and only if $M_R=0$ and there are no nonzero absolute left zero divisors (i.e. elements x such that $xA=0$) in A^n for every n.*

PROOF. In one direction the proof is clear. Assume then that $M_R=0$ and that there are no nonzero absolute zero divisors in A^n. By DCC we can write $A=e_1A+\cdots+e_nA+N_0$ where the e_iA are indecomposable right ideals and N_0 is nilpotent. If $e_1 \neq 0$ then it is not in M_R. Then there must exist an x' in A such that $e_1x' \neq 0$ and such that e_1x' is not in $e_1x'A$. Else e_1 is in M_R by Theorem 1. Thus $e_1x'A \neq e_1A$. Since e_1A is indecomposable, $e_1x'A$ must be nilpotent. Since $e_1x' = e_1^{n-1}x'$ is in A^n for every n, $e_1x'A$ cannot be zero. Let N be the max. nilpotent ideal of A. Then $e_1N > e_1e_1x'A = e_1x'A \neq 0$. However the chain $e_1N > e_1N^2 > \cdots > e_1N^m > \cdots$ terminates in zero after a finite number of steps. Then there exists an integer $w \geqq 1$ such that $e_1N^w \neq 0$, $e_1N^{w+1} = 0$. Let x'' be an element of N^w such that $e_1x'' \neq 0$ and let $x = e_1x''$. Then $xN = 0$, $x \neq 0$. Since $x = e_1x'' = e_1^{n-1}x''$ is in A^n for every n, $xA \neq 0$. Since $xN_0 < xN = 0$, there must exist an e_i such that $xe_iA \neq 0$. However since $xN = 0$, xe_iA is a minimal right ideal of A. For if $0 \neq I \leqq xe_iA$, where I is a right ideal of A, let $Q = \{y \text{ in } e_iA : xy \text{ is in } I\}$. Then Q is a right ideal of A, $Q \leqq e_iA$. If $Q \neq e_iA$, then Q is nilpotent because e_iA is indecomposable. Then $xQ < xN = 0$ and therefore if z is in I, $z = xe_iz'$ for some z' in A, e_iz' is in Q, $xe_iz' = 0$, $I = 0$. Thus $Q = e_iA$, $I = xe_iA$.

Finally let y' be an element of A such that $xe_iy' \neq 0$, and let $y = e_iy'$. Then xy is in xe_iA, $xyN = 0$, $xyA \neq 0$ (since $xy = e_i^{n-2}xy$ is in A^n for every n). Now $xyA \leqq xe_iA$ and since xe_iA is minimal, $xyA = xe_iA$. Therefore xy is in xyA. Furthermore if $xyu \neq 0$, $xyuA \neq 0$ (again since $xyu = e_i^{n-3}xyu$ is in A^n for every n) and thus xyu is in $xyuA$. Therefore xy is in M_R by Theorem 1. This is a contradiction and thus all the e_i are zero, and A must be nilpotent. Q.E.D.

Corollary 1. *If A has DCC on left ideals then A is nilpotent if and only if $M_L = 0$ and there are no nonzero absolute right zerodivisors in A^n for every n.*

Since M_R contains the max. regular ideal $\overline{M}$ and when $A - J$ is regular, or in particular when A has DCC on right ideals, then $\overline{M} = 0$ if and only if A is bound to J, Theorem 6, [**4**], we can conclude that when $M_R = 0$ and A has DCC on right ideals, A is bound to J. Combining this with Theorem 5 we have

Corollary 2. *If A has DCC on right ideals and $M_R = 0$, then either A is nilpotent or A is bound to N and A has an absolute left zero divisor in A^n for every n.*

The converse is also true. Using the fact that $M_R(A - M_R) = 0$ we have

Corollary 3. *If A has DCC on right ideals, then $A - M_R$ is nilpotent if and only if $xA \leqq M_R$ and x in $(A - M_R)^n$ for every n, implies that x is in M_R.*

When $M_R = M_L = 0$ the zero divisor condition can be slightly weakened.

THEOREM 6. *If A has DCC on one-sided ideals, then A is nilpotent if and only if $M_R = M_L = 0$ and there are no nonzero total divisors of zero in A^n for every n, i.e. elements x such that $xA = Ax = 0$ and x in A^n for every n.*

PROOF. In one direction the proof is clear. Conversely, we write as before $A = e_1A + \cdots + e_nA + N_0$ where the e_iA are indecomposable right ideals and N_0 is nilpotent. If $e_1 \neq 0$, consider the chain $e_1N > e_1N^2 > \cdots > e_1N^\alpha = 0$. There exists an integer γ such that $e_1N^\gamma \neq 0$, $e_1N^{\gamma+1} = 0$. If $\gamma = 0$, let x be any element such that $e_1x \neq 0$. Consider $e_1xA \leqq e_1A$. If $e_1xA \neq e_1A$, then since e_1A is indecomposable, e_1xA is nilpotent, e_1xA is in N. Then $e_1 \cdot e_1xA < e_1N = 0$. Then e_1x is properly nilpotent, e_1x is in N. Then $e_1 \cdot e_1x = 0 = e_1x$, a contradiction. On the other hand if $e_1xA = e_1A$ then e_1x is in e_1xA and since e_1 is in e_1A, e_1 is in M_R by Theorem 1. Then $e_1 = 0$, a contradiction. Thus $e_1N \neq 0$, $\gamma \geqq 1$. Similarly there exists an integer $\rho \geqq 1$ such that $N^\rho e_1 \neq 0$, $N^{\rho+1}e_1 = 0$. In this way we obtain for each e_i, integers γ_i and ρ_i such that $e_iN^{\gamma_i} \neq 0$, $e_iN^{\gamma_i+1} = 0$, $N^{\rho_i}e_i \neq 0$, $N^{\rho_i+1}e_i = 0$. Let β be the maximum of the γ_i and ρ_i. Then setting $e = e_1 + \cdots + e_n$, $eN^{\beta+1} = N^{\beta+1}e = 0$, and either eN^β or $N^\beta e \neq 0$. Suppose $eN^\beta \neq 0$. Then for some e_j, $e_jN^\beta \neq 0$. Take x' in N^β such that $e_jx' \neq 0$ and let $x = e_jx' = e_jx = ex$. Then $xN = 0$. Since $M_L = 0$ and x is in Ax, there must exist (Theorem 1) an element y in A such that $yx \neq 0$ and such that yx is not in Ayx. We may take $y = ye_j = ye$. The element y must be in N. For yx not in Ayx implies y not in $Ay = Aye_j$. Thus $Aye_j \neq Ae_j$ and since Ae_j is indecomposable, Aye_j is nilpotent. Then $ye_j \cdot ye_j$ is nilpotent, ye_j is nilpotent and clearly $ye_j = y$ is properly nilpotent and therefore y is in N. Then yx is in $Ne_jN^\beta \leqq N^{\beta+1}$ and therefore $yxe = 0$. Also $yxN = 0$ and thus $yxA = 0$. Also $eyx = 0$, since $eN^{\beta+1} = 0$. Note that $yx = ye_jx = ye_j^{n-2}x$ is in A^n for every n. If yx is not a total zerodivisor, $Ayx \neq 0$. Then, $Nyx \neq 0$, since $eyx = 0$. Let y_1 be an element in N such that $y_1yx \neq 0$. As above $y_1yxA = ey_1yx = 0$. If y_1yx is not a total divisor of zero, $Ny_1yx \neq 0$. We continue this process until $t = y_{\beta-1} \cdots y_1yx \neq 0$, $tA = 0$, $et = 0$. Then t is in $N^\beta e_jN^\beta$ and Nt is in $N^{\beta+1}e_jN^\beta = 0$. Thus $At = tA = 0$, and $t = y_{\beta-1} \cdots y_1ye_j^mx$ is in A^n for every n. This is impossible and thus $e_i = 0$ for every i, $A = N_0$, A is nilpotent. Q.E.D.

The Hopkins example mentioned earlier shows that the divisor of zero restrictions cannot be removed, for $M_R = 0$, A has DCC and is not nilpotent. To obtain an example for Theorem 6, let A be an algebra of dimension 4 over a field F, with basal elements e, u, v, w and the following multiplication table:

	e	u	v	w
e	e	0	v	0
u	u	0	w	0
v	0	0	0	0
w	0	0	0	0

Then $M_R = M_L = 0$ and A is not nilpotent. The radical is generated by u, v, and w. The element w is a total divisor of zero and in A^n for every n. This algebra is in fact subdirectly irreducible with minimal ideal generated by w. For let I be any ideal of A, with $x = \alpha e + \beta u + \gamma v + \delta w$ in I. Then $xe = \alpha e + \beta u$, $ex = \alpha e + \gamma v$, $xv = \alpha v$, $ux = \alpha u$. Then if $\alpha \neq 0$, I contains u, v and then w, and also e, $I = A$. If $\alpha = 0$, I contains βu and γv and therefore δw. If $\beta \neq 0$, then u and w are in I. If $\gamma \neq 0$, v and w are in I. If $\beta = \gamma = 0$, $I = (w)$. Thus there are precisely five nonzero ideals: (w), (u, v), (v, w), (u, v, w), (u, v, w, e). Since w is not in M_R or M_L they are zero.

We now return to the commutative case but drop DCC. Then $M = \{x : x \text{ is in } xA\}$ and thus M contains all idempotents. In (1) all $M_i' =$ the corresponding M_i.

THEOREM 7. *If A is commutative then $M = 0$ if and only if A is isomorphic to a subdirect sum of subdirectly irreducible rings with an absolute divisor of zero in their minimal ideals. That is, they are of type β* [7].

From [7] we know that a commutative subdirectly irreducible ring with the ascending chain condition is either nilpotent or has a unity element. Thus we have

THEOREM 8. *If A is commutative with ACC then $M = 0$ if and only if A is isomorphic to a subdirect sum of nilpotent subdirectly irreducible rings.*

Though Theorems 7 and 8 seem to yield radical-like results, this may be misleading. Let A be the ring of even integers. It has ACC but not DCC. Also $M = 0$ and A is isomorphic to a subdirect sum of nilpotent rings, namely

$$A = (A/(4), A/(8), \cdots, A/(2^n), \cdots)$$

where $A/(2^n)$ has 2^{n-1} as an absolute divisor of zero and is nilpotent for every n. However $J = 0$ and therefore A is isomorphic to a subdirect sum of fields, namely

$$A = (A/(6), A/(10), A/(14), \cdots, A/(2p), \cdots)$$

where p is a prime. Thus A may be semi-simple and still have $M=0$.

This example also shows that DCC is necessary to obtain nilpotence. To see that ACC is necessary to obtain a subdirect sum of nilpotent rings, we may consider the example in [**7**] which is commutative, subdirectly irreducible, has neither chain condition, has $M=0$ and is not nilpotent.

3. **The cases** $A=M_R$, $A=M_L$, $A=M_R=M_L$. In studying the existence of right, left and two sided unities, Baer [**1**; **2**; **3**], concerned himself to some extent with right and left D-regularity. We summarize some of his results in the language of M_R, M_L and M:

With DCC on one-sided ideals:

1a. $A=M_R$ if and only if A has a right unity.

2a. $A=M_R=M_L$ if and only if A has a unity.

3a. A commutative, $A=M$ if and only if A has a unity.

If $A-J$ has a unity or if $A-J$ has DCC on one-sided ideals:

2b. $A=M_R=M_L$ if and only if A has a unity.

3b. A commutative, $A=M$ if and only if A has a unity.

However 1a needed some strengthening:

1b. A has a right unity if and only if $A=M_R$ and when $A=J+Ax$, x must be in Ax.

Baer also proved, for a ring with DCC on one-sided ideals:

c. A has a right unity if and only if A has a non-right-zero divisor, i.e. an element x such that $yx=0$ implies $y=0$.

Let A be a ring with ACC on left ideals. Then, as is well known, every left ideal of A is finitely generated and in particular $M_R = \{\sum_1^m n_i a_i + x_i a_i\}$, x_i in A, a_i in M_R, n_i integers. By the corollary to Lemma 5, there exists an element e in A such that $M_R=M_R e$, pointwise. If we assume $M_R=A$, e is a right unity element.

THEOREM 9. *If A has ACC on left ideals, then A has a right unity if and only if $A=M_R$.*

Passing now to rings without chain conditions, we first prove

LEMMA 6. *A has a left unity if and only if there exists an element x in A such that x is in xA and x is not a left zero divisor.*

PROOF. If A has a left unity f, then f is in fA, and if $fy=0$ then clearly $y=0$. Conversely if x is in xA, $x=xe$, then for every y, $x(y-ey)=0$ and since x is not a left zero divisor, $y=ey$, e is a left unity.

Note that if x were also not a right zero divisor, then A would have a unity. For $x=ex=xe$ and $(y-ye)x=0$ yields $y=ye$ for every y.

COROLLARY. *A has a unity if and only if either M_R or M_L has a non-zero-divisor.*

We thus have for rings without chain conditions:

THEOREM 10.

a. *If $A = M_R$, then A has a left unity if and only if A has a non-left-zero-divisor; A has a unity if and only if A has a non-zero-divisor.*

b. *If $A = M_L$, A has a right unity if and only if A has a non-right-zero-divisor.*

c. *If $A = M_R = M_L$, then A has a unity if and only if A has a non-zero-divisor if and only if A has a non-right and a non-left-zero-divisor.*

Note that then if $A = M_R = M_L$ and if A has neither a right nor a left unity, then every element of A is a two-sided divisor of zero.

In the commutative case, when $A = M$ it is thus clear that A has a unity if and only if A has a non-zero-divisor. We can obtain more. For when A is expressed as a subdirect sum of subdirectly irreducible rings A_i, each A_i must be equal to its M_i: Let x_i be any element of A_i. It must appear in the expansion of some element, say $x = (x_1, \cdots, x_i, \cdots)$. Since $A = M$, there exists an element y in A such that $xy = x$. Let $y = (y_1, \cdots, y_i, \cdots)$. Then $x_i y_i = x_i$ and x_i is in M_i, $A_i = M_i$. However from [7], it is clear that if a commutative subdirectly irreducible ring is equal to its maximal D-regular ideal, it has a unity element. In fact it is either a field or has a unity and is a field modulo its set (an ideal) of zero-divisors. If in addition it has ACC, it is either a field or has a unity and is a field modulo its maximal nilideal. Thus we have:

THEOREM 11. *If A is a commutative ring and $A = M$, then A is isomorphic to a subdirect sum of subdirectly irreducible rings each with a unity. Some are fields and others are fields modulo the ideal of zero-divisors. If A has ACC, the latter set are fields modulo their maximal nilideals.*

Of course A itself may not have a unity, for let A be the weak direct sum of an infinite number of fields. Every element of A is a zero-divisor and A has no unity.

In summary we have:

$A = M_R = M_L$ if and only if A has a unity and any one of the five following conditions: DCC on one-sided ideals; ACC on one-sided ideals; a unity element in $A - J$; a non-zero-divisor; a non-left and non-right-zero-divisor.

$A = M_R$ if and only if A has a right unity and one of the following three conditions: DCC on right ideals; ACC on left ideals; $A - J$ has a unity element and $A = J + Ax$ implies that x must be in Ax.

$A = M_R$ implies that A has a left unity, if it has a non-left-zero-divisor.

$A = M_R$ if and only if A has a unity, if A has a non-zero-divisor.

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