ON A PROBLEM OF D. R. HUGHES

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D. R. Hughes (Bull. Amer. Math. Soc. vol. 63 (1957) p. 209) has proposed the following problem:

Let \( G \) be a group and \( p \) a prime. Define \( H_p(G) \) to be the [normal] subgroup of \( G \) generated by all the elements of \( G \) which do not have order \( p \).

Is the following conjecture true: either \( H_p(G) = \{1\} \) or \( H_p(G) = G \) or \([G: H_p(G)] = p\)?

He remarks that the conjecture is true for \( p = 2 \).

In this note we prove Hughes' conjecture for \( p = 3 \).

We shall use the following notation: if \( h, g_1, \ldots, g_n \in G \) and \( a_1, \ldots, a_n \) are integers then

\[
h^{a_1g_1 + \cdots + a_ng_n} = g_1^{-1}h^{-1}g_1^{-1}h^{-1}g_2^{-1}h^{-1}g_2^{-1}h^{-1}g_n^{-1}h^{-1}g_n.
\]

**Lemma 1.** If \( h \in H_p, x \in H_p \) then

\[
h^{1+x+x^2+\cdots+x^{p-1}} = 1.
\]

**Proof.** Since \( x^{p-1} \in H_p \), all elements of \( H_p x^{p-1} \) have order \( p \). In particular

\[
1 = (hx^{p-1})^p = hx^{p-1} \cdot hx^{p-1} \cdots hx^{p-1} = h \cdot x^{-1}hx \cdot x^{-2}hx^2 \cdots x^{-(p-1)}hx^{p-1} = h^{1+x+\cdots+x^{p-1}}.
\]

**Lemma 2.** If \( h \in H_3 \) and \( xH_3 \neq yH_3 \) then \( h^x+y = h^y+z \).

**Proof.** By hypothesis \( z = x^{-1}y \in H_3 \). Hence by Lemma 1

\[
1 = h_1^{1+z+z^2} = h_1^{1+z^2+z}, \quad h_1 \in H_3
\]

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or $h_1^{x^2} = h_1^{x^2 + z}$. Setting $h_1 = h^{x^2}$ proves the lemma.

**Theorem.** If $[G: H_3] > 3$ then $H_3 = \{1\}$.

**Proof.** Since all elements of $G/H_3$ are of order 3, every finitely generated subgroup of $G/H_3$ is finite by Burnside's theorem. In particular, therefore, $G/H_3$ has an Abelian subgroup of order 9. Let such a subgroup be generated by $xH_3, yH_3$. We have

\[
3 = - (1 + x + x^2)y - y^2(1 + x + x^3) + (1 + y + y^2) \\
+ (1 + xy + y^2x) + (1 + x^2y + y^2x) = f(x, y).
\]

But according to Lemma 2 we have therefore $h^3 = h^{f(x, y)}$ for every $h \in H_3$, and by Lemma 1 this implies $h^3 = 1$. In other words $H_3$ contains only the identity.

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