ON THE HILBERT MATRIX, I

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1. For fixed $k < 1$ the generalized Hilbert matrix is $H_k = ((m+n+1-k)^{-1})$, $m$, $n=0$, 1, 2, \ldots. By a latent root of $H_k$ we mean a complex number $\lambda$ such that there exists a non-null sequence of complex numbers $\{x_n\}_{n=0}^{\infty}$ with the property that

$$\sum_{n=0}^{\infty} (n + m + 1 - k)^{-1}x_n$$

converges to $\lambda x_m$ for all non-negative integers $m$. It is known (see [6; 3], and [4]) that $\pi \csc \pi k$ is a latent root of $H_k$ if $k > 0$. Taussky [9] posed the problem of determining whether $\pi$ is a latent root of $H_0$. This problem was solved by Kato [5], who applied a general theory to show that $H_k$ has the latent root $\pi$ when $1/2 \geq k$.

We shall prove

**THEOREM 1.** Every complex number with positive real part is a latent root of $H_k$. 

2. The Whittaker function $W_{k,m}$ is defined in [11, p. 340] by

$$\Gamma\left(m - k + \frac{1}{2}\right) W_{k,m}(x)x^{-m-1/2}$$

$$= \int_{1/2}^{\infty} e^{-zs} \left(s + \frac{1}{2}\right)^{k+m-1/2} \left(s - \frac{1}{2}\right)^{m-k-1/2} ds,$$

where $k<1/2 + \Re m$ and $\Gamma$ is the gamma function. For $n=0$, 1, 2, \ldots, let $\phi_n(x) = e^{-x/2}L_n(x)$, where $L_n$ is the $n$th Laguerre polynomial normalized so that the $L^2(0, \infty)$ inner product

$$(\phi_n, \phi_m) = \int_0^{\infty} e^{-t}L_n(t)L_m(t)dt = \delta_{n,m}.$$

If $x \geq 0$

$$\int_0^{\infty} e^{-tx}\phi_n(t)dt = \left(x - \frac{1}{2}\right)^n\left(x + \frac{1}{2}\right)^{-n-1}$$

and $|\phi_n(x)| \leq 1$ [8, p. 159].

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We define the operator $\mathcal{K}_k$ by
\[(2.3) \quad (\mathcal{K}_k f)(x) = \Gamma(1 - k) \int_0^\infty W_{k,1/2}(x + t)(x + t)^{-1}f(t)dt.\]

By 2.1, 2.2, and the Fubini theorem, if $x > 0$, then
\[
(\mathcal{K}_k\phi_n)(x) = \int_0^\infty \int_{1/2}^\infty \left(s + \frac{1}{2}\right)^k\left(s - \frac{1}{2}\right)^{-k}e^{-s(x+t)}ds\phi_n(t)dt
\]
\[
= \int_{1/2}^\infty e^{-xs}\left(s - \frac{1}{2}\right)^{n-k}\left(s + \frac{1}{2}\right)^{k-n-1}ds
\]
\[
= \Gamma(1 - k + n)W_{k-n-1/2,0}(x)x^{-1/2},
\]
and by 2.4 and 2.2,
\[
(\mathcal{K}_k\phi_n, \phi_m) = \int_0^\infty \int_{1/2}^\infty e^{-sx}\left(s - \frac{1}{2}\right)^{n-k}\left(s + \frac{1}{2}\right)^{k-n-1}ds\phi_n(x)dx
\]
\[
= \int_{1/2}^\infty \left(s - \frac{1}{2}\right)^{n+m-k}\left(s + \frac{1}{2}\right)^{k-n-m-2}ds
\]
\[
= (n + m + 1 - k)^{-1}.
\]

Thus if we consider $\mathcal{K}_k$ as an operator on $L^2(0, \infty)$, then $H_k$ is the matrix representation of $\mathcal{K}_k$ relative to the complete orthonormal set $\{\phi_n\}$. Henceforth we shall take $u$ to be a complex number such that $-1/2 < \Re u < 1/2$, $k < 1$, and $f(x) = W_{k,u}(x)x^{-1}$. The equation
\[(2.6) \quad \pi \sec \pi u f(x) = (\mathcal{K}_k f)(x)\]
is a particularization of an equation noted by Hari Shanker [7]. Hence a reasonable candidate for a solution $\{x_n\}$ of the matrix equation
\[(2.7) \quad \sum_{n=0}^\infty (n + m + 1 - k)^{-1}x_n = \pi \sec \pi ux_n\]
is given by
\[(2.8) \quad x_n = \int_0^\infty f(t)\phi_n(t)dt.\]

In the remainder of this note we shall show that indeed the $\{x_n\}$ defined by (2.8) satisfy (2.7).

3. From [1, Chapter 6], we know that $f(x) = O(x^{-1/2 - |\Re u|})$ and $g(x) = W_{k,-n-1/2,0}(x)x^{-1/2} = O(\log x)$ as $x \to 0$, and $f(x) = O(e^{-x/2}x^k)$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ g(x) = O(e^{-x/2}) \] as \( x \to \infty \). It follows from these estimates that \( f \in L(0, \infty) \) so

\[ |x_n| \leq \int_0^\infty |f(t)\phi_n(t)| \, dt \leq \int_0^\infty |f(t)| \, dt < \infty, \]

and the \( x_n \) are uniformly bounded. Also, the integrals in the following calculation are absolutely convergent so we may freely change the orders of integration. From (2.6), (2.8), and (2.4),

\[
\pi \sec \pi u x_m = \pi \sec \pi u \int_0^\infty f(x)\phi_m(x) \, dx
\]

(3.1)

\[ \int_0^\infty (3C_kf)(x)\phi_m(x) \, dx = \int_0^\infty f(x)(3C_k\phi_m)(x) \, dx \]

\[ = \int_{1/2}^\infty \int_0^\infty e^{-sx}f(x) \, dx \left( s - \frac{1}{2} \right)^{m-k} \left( s + \frac{1}{2} \right)^{k-m-1} \, ds. \]

Put \( z = (s-1/2)/(s+1/2)^{-1} \), so \( s = 2^{-1}(1+z)(1-z)^{-1} \) and

\[ \pi \sec \pi u x_m = \lim_{T \to 1^-} \int_0^T \int_0^\infty \exp \left[ -\frac{1}{2} x(1+z)(1-z)^{-1} \right] f(x)z^m(1-z)^{-1} \, dz. \]

But [8, p. 97]

\[ \exp \left[ -\frac{1}{2} x(1+z)(1-z)^{-1} \right](1-z)^{-1} = \sum_{n=0}^\infty z^n\phi_n(x), \]

where the series converges uniformly in \( x \) and \( z \) for \( 0 \leq x < \infty, 0 \leq z \leq T < 1 \). Hence

\[
\pi \sec \pi u x_m = \lim_{T \to 1^-} \int_0^T \sum_{n=0}^\infty x_n z^{n+m-k} \, dz
\]

\[ = \lim_{T \to 1^-} \sum_{n=0}^\infty x_n \int_0^T z^{n+m-k} \, dz \]

\[ = \lim_{T \to 1^-} \sum_{n=0}^\infty \frac{(n+m+1-k)^{-1}}{T^n+m+1-k} x_n T^{n+m+1-k} \]

\[ = \lim_{T \to 1^-} \sum_{n=0}^\infty \frac{(n+m+1-k)^{-1} x_n T^n.} \]

Since the \( x_n \) are uniformly bounded we may apply the Littlewood
Tauberian theorem [10, p. 233] to this last expression and infer that (2.7) is true. Finally, \( w = \pi \sec \pi u \) maps the strip \(-1/2 < \Re u < 1/2\) onto the open half-plane \( 0 < \Re w \), so the proof of Theorem 1 is complete.

4. If we suppose \( k - 1/2 < u < 1/2, u \geq 0 \), then by (2.1) and (3.1), \( f(x), (x > 0) \), and \( x_0, x_1, x_2, \cdots \), are positive. Upon setting \( \lambda = \pi \sec \pi u \) we have

**Theorem 2.** If \( k < 1/2 \) and \( \lambda \geq \pi \), or if \( 1 > k \geq 1/2 \) and \( \lambda > \pi \csc \pi k \), then there exists a positive root vector \( \{x_n\} \) corresponding to the latent root \( \lambda \) of \( H_k \).

This theorem furnishes a solution to a problem posed by Kato in [5, p. 80].

5. I am indebted to the referee for

**Theorem 3.** Consider \( H_k \) as a linear operator on the sequential Banach space \( l^q \), where \( 2 < q < \infty \). Then \( H_k \) is bounded and \( \pi \sec \pi u \) is an eigenvalue of \( H_k \) whenever \( | \Re u | < 1/2 - 1/q \).

**Proof.** The boundedness of \( H_k \) follows from [2, Theorem 364, p. 258]. The restriction on \( \Re u \) guarantees that \( f \in L^p(0, \infty) \), where \( p^{-1} + q^{-1} = 1 \). Since the \( \phi_n \) are uniformly bounded it follows from F. Riesz's extension of the Hausdorff-Young theorem [12, p. 191] that \( \{x_n\} \) given by (2.8) belongs to \( l^q \). Finally, by 2.7, \( \pi \sec \pi u \) is an eigenvalue of \( H_k \).

**References**


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