ON THE GIBBS' PHENOMENON IN A CERTAIN EIGENFUNCTION SERIES

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I. Introduction. In a paper by B. Friedman and Luna Mishoe [1] it is shown that the series,

$$
\sum_{n=-\infty}^{\infty} a_n u_n(x);
$$

where

$$
a_n = \left( \int_{0}^{1} u_n(\xi) B^* v_n(\xi) d\xi \right)^{-1} \int_{0}^{1} f(\xi) B^* v_n(\xi) d\xi; \quad u_n(x)
$$

being eigenfunctions or nonzero solutions of the system

$$
u'' + q(x)u + \lambda[p(x)u - u'] = 0, \quad u(0) = u(1) = 0
$$

and $v_n(x)$ being eigenfunctions of the system adjoint to (2), where $B^* = d/dx + p(x)$; converges to;

$$
1/2[f(x+) + f(x-)] + c \exp \int_{0}^{x} p dt;
$$

provided $f(x)$ is of bounded variation, and $p(x)$ possesses a continuous second derivative, and $q(x)$ is continuous on $(0, 1)$. In the expression (3) above

$$
c = -1/2 \left[ f(0+) + f(1-) \exp - \int_{0}^{1} p dt \right].
$$

In this paper the following theorem is proved:

THEOREM. The partial sum $\sigma_n(x) = \sum_{k=-n}^{n} a_k u_k(x)$ exhibits Gibbs' phenomenon for the function $g(x) = f(x) + c \exp \int_{0}^{x} p dt$ whenever the Fourier partial sum $S_n(x) = \sum_{k=-n}^{n} \exp (2k\pi i) \int_{0}^{1} f(\xi) \exp (-2k\pi i\xi) d\xi$ exhibits Gibbs' phenomenon for $f(x)$.

II. A relationship between $\sigma_n(x)$ and $S_n(x)$. In [1] it is shown that:

$$
u_n(x) = \lambda_n^{-1} \left[ \exp \left( \lambda_n x - \int_{0}^{x} p dt \right) - \exp \left( \int_{0}^{x} p dt \right) \right] + O\left( \frac{1}{\lambda_n^2} \right),
$$

$$
B^* v_n(x) = \exp \left( -\lambda_n x + \int_{0}^{x} p dt \right) + O\left( \frac{e^{-\lambda_n x}}{\lambda_n} \right).
$$

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where $O(1)$ means a bounded function as $|\lambda| \to \infty$.

In [2] it is shown that:

\begin{equation}
\lambda_n = 2n\pi i + \int_0^1 pdt + O\left(\frac{1}{n}\right).
\end{equation}

From (5) and (6) it follows that

\begin{equation}
\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi = \int_0^1 \frac{1}{\lambda} d\xi + \int_0^1 O\left(\frac{1}{\lambda}\right) d\xi - \int_0^1 \exp\left(-\lambda_n \xi + \int_0^\xi pdt\right) d\xi = \frac{1 + O(1/\lambda_n)}{\lambda_n}
\end{equation}

and consequently

\begin{equation}
\left[\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi\right]^{-1} = \frac{\lambda_n}{1 + O(1/n)} = \lambda_n \left[1 + O\left(\frac{1}{n}\right)\right]
\end{equation}

for large $n$.

Now in (5) if we use the fact that $\exp\left(-\int_0^\xi pdt\right) = 1 + O(1)$ we have

\begin{equation}
u_n(x) = \lambda_n^{-1}\left[e^{\lambda_n x} + O(1) + O\left(\frac{1}{n}\right)\right]
\end{equation}

and

\begin{equation}
\int_0^1 f(\xi)B^*v_n(\xi)d\xi = \int_0^1 f(\xi)e^{-\lambda_n \xi}d\xi + O(1).
\end{equation}

From (8), (9), and (10) we obtain

\begin{equation}
a_n u_n(x) = u_n(x)\left[\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi\right]^{-1}\int_0^1 f(\xi)B^*v_n(\xi)d\xi = e^{\lambda_n x}\int_0^1 f(\xi)e^{-\lambda_n \xi}d\xi + O(1).
\end{equation}

Using the value of $\lambda_n$ as given in (6) we have

\begin{equation}
\lambda_n = e^{2n\pi i x}\cdot \exp\left(2x\int_0^1 pdt + O\left(\frac{x}{n}\right)\right) = e^{2n\pi i x} + O(1)
\end{equation}

and consequently

\begin{equation}
a_n u_n(x) = e^{2n\pi i x}\int_0^1 f(\xi)e^{-2n\pi i \xi}d\xi + E_n(x),
\end{equation}

where
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\[ E_n(x) = O(1) + O(1/n). \]

And we have,

\[ \sigma_n(x) = \sum_{k=-n}^{n} a_k u_k(x) = \sum_{k=-n}^{n} e^{2k\pi ix} \int_{0}^{1} f(\xi) e^{-2k\pi i\xi} d\xi + \sum_{k=-n}^{n} E_k(x) \]

or \( \sigma_n(x) = S_n(x) + T_n(x) \) where \( S_n(x) \) is the Fourier partial sum of \( f(x) \) and \( T_n(x) = \sum_{k=-n}^{n} E_k(x) \). This is our desired relationship.

III. Uniform convergence of the series \( \sum_{n=-\infty}^{\infty} E_n(x) \) in \((0, 1)\).

Since \( \sum_{n=-\infty}^{\infty} a_n u_n(x) = 1/2[f(x+) + f(x-)]/2 + c \exp \int_{0}^{x} p dt \); it follows that

\[ \sum_{n=-\infty}^{\infty} \left[ e^{2n\pi ix} \int_{0}^{1} f(\xi) e^{-2n\pi i\xi} d\xi + E_n(x) \right] \]

\[ = 1/2[f(x+) + f(x-)] + c \exp \left( \int_{0}^{x} p dt \right) \]

and since

\[ \sum_{n=-\infty}^{\infty} e^{2n\pi ix} \int_{0}^{1} f(\xi) e^{-2n\pi i\xi} d\xi = 1/2[f(x+) + f(x-)] \quad \text{for } 0 < x < 1 \]

we have

\[ \sum_{n=-\infty}^{\infty} \left[ e^{2n\pi ix} \int_{0}^{1} f(\xi) e^{-2n\pi i\xi} d\xi + E_n(x) \right] \]

\[ = c \exp \left( \int_{0}^{x} p dt \right) + \sum_{n=-\infty}^{\infty} e^{2n\pi ix} \int_{0}^{1} f(\xi) e^{-2n\pi i\xi} d\xi \]

and it follows that

\[ \sum_{n=-\infty}^{\infty} E_n(x) = c \exp \left( \int_{0}^{x} p dt \right). \]

Now since \( p(x) \) possess a continuous second derivative on \((0, 1)\) it follows that \( c \exp \left( \int_{0}^{x} p dt \right) \) is both continuous and of bounded variation on \((0, 1)\) and therefore may be expanded into a Fourier series which converges uniformly to \( c \exp \left( \int_{0}^{x} p dt \right) \) for \( 0 < x < 1 \). From this we conclude that

\[ \sum_{n=-\infty}^{\infty} E_n(x) = c \exp \left( \int_{0}^{x} p dt \right) \quad \text{uniformly for } 0 < x < 1. \]
IV. Gibbs' Phenomenon. Let \( f(x) \) possess a finite discontinuity at \( x = a_1 \) and let \( \alpha_n \) and \( \beta_n \) be the abscissa and ordinate respectively of the first maxima (or minima) of \( y = S_n(x) \) to the left or right of \( x = a_1 \), then as

\[
\lim_{n \to \infty} \alpha_n = a_1.
\]

But on the left of \( a_1 \)

\[
\beta_n \to f(a_1 - 0) + \left[ \frac{f(a_1 + 0) - f(a_1 - 0)}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx \right]
\]

and on the right of \( x = a_1 \)

\[
\beta_n \to f(a_1 + 0) \left[ \frac{f(a_1 + 0) - f(a_1 - 0)}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx \right].
\]

This is Gibbs' Phenomenon for the partial sum \( S_n(x) \) of the Fourier Series of \( f(x) \).

Since \( T_n(x) = c \exp \left( \int_0^x \frac{1}{p} \, dt \right) \) uniformly in \((0, 1)\) as \( n \to \infty \), this partial sum does not exhibit Gibbs' Phenomenon.

Note that corresponding to the maxima (or minima) \((\alpha_n, \beta_n)\) on the curve, \( y = S_n(x) \) there is a point \((\alpha_n, \beta_1)\) on the curve \( y^1 = \sigma_n(x) \) such that

\[
\beta_1 = \beta_n + c \exp \left( \int_0^{\alpha_n} \frac{1}{p} \, dt \right).
\]

Now as \( n \to \infty \), \( \alpha_n \to a_1 \)

\[
\beta_1 \to f(a_1 - 0) + \left[ \frac{f(a_1 + 0) - f(a_1 - 0)}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx \right] + c \exp \left( \int_0^{a_1} \frac{1}{p} \, dt \right)
\]

on the left of \( x = a_1 \) and

\[
\beta_1 \to f(a_1 + 0) \left[ \frac{f(a_1 + 0) - f(a_1 - 0)}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx \right] + c \exp \left( \int_0^{a_1} \frac{1}{p} \, dt \right)
\]

on the right of \( x = a_1 \). Or as \( n \to \infty \)

\[
\lim_{n \to \infty} \alpha_n = a_1.
\]
and since $f(a_1 + 0) - f(a_1 - 0) = g(a_1 + 0) - g(a_1 - 0)$ where

$$g(x) = f(x) + c \exp \left( \int_0^x \rho \, dt \right)$$

we have

(21) \[ \beta_n \to g(a_1 - 0) + \left[ \frac{g(a_1 + 0) - g(a_1 - 0)}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \right] \]

on the left of $x = a_1$ and

(22) \[ \beta_n \to g(a + 0) - \left[ \frac{g(a_1 + 0) - g(a_1 - 0)}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \right] \]

on the right of $x = a_1$, where $g(x) = f(x) + c \exp (\int_0^x \rho \, dt)$. The results in (20), (21), and (22) shows clearly that $y = \sigma_n(x)$ exhibits Gibbs' Phenomenon for

$$g(x) = f(x) + c \exp \left( \int_0^x \rho \, dt \right)$$

wherever $y = S_n(x)$ exhibits this Phenomenon for $f(x)$.

References


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