1. Introduction. We consider below the Cauchy problem for systems of homogeneous linear partial differential equations (DE's) with constant coefficients. That is, we ask for solutions of
\[
\frac{\partial u_j}{\partial t} = \sum_{j, k = 1, \ldots, n} p_{jk}(D_1, \ldots, D_r) u_k
\]
satisfying \( u(x; 0) = v_0(x) \), where \( v_0(x) \) is supposed given in infinite \((x_1, \ldots, x_r)\)-space. Here \( \|p_{jk}(D)\| \) denotes a general matrix of polynomials with constant coefficients in the differential operators \( D_i = \partial / \partial x_i \).

Our main result is an existence theorem for the Cauchy problem in infinite space, for any system (1) which is regular in the following sense [1]. For each real wave-vector \( q = (q_1, \ldots, q_r) \), we let \( \lambda_1(q), \ldots, \lambda_n(q) \) denote the characteristic values of the \( q \)-matrix \( \|p_{jk}(i q)\|, i = (-1)^{1/2} \). We define the spectral norm of (1) as
\[
\sigma[P] = \sup_{q, j} \Re \{\lambda_j(q)\} = \sup_q \sigma[P(q)],
\]
and call the system (1) regular when \( \sigma[P] < +\infty \).

In this existence theorem, the solutions are defined as the semi-orbits, for \( t > 0 \), of a \( C_0 \)-semigroup acting on an appropriate Banach space. Thus, in particular, we solve a family of "abstract Cauchy problems" ([5; 6]), each corresponding in some sense to the system (1). Further, it is shown that the \( C_0 \)-semigroup can be so constructed that all semi-orbits represent literal solutions of (1). Finally, in the "hyperbolic" case, a \( C_0 \)-group can be constructed.

Though similar methods have been used by Gelfand and Šilov ([3; 4]), the representation in terms of \( C_0 \)-semigroups is new. The specific results of Hille [5] are for quite special DE's. Related results have also been obtained by L. Schwartz [7] and, using other methods, by L. Gårding (Acta Math. vol. 85 (1951) pp. 1-62) and A. Lax (Communications on Pure and Applied Mathematics, vol. 9 (1956) pp. 135-70).

2. The Fréchet space \( \Phi \). For any fixed wave-vector \( q = (q_1, \ldots, q_r) \), any system (1) has solutions of the special form
(3) \[ u_j(x; t) = f_j(q; t)e^{iq \cdot x}, \quad q \cdot x = q_1x_1 + \cdots + q_rx_r. \]

Namely, it is sufficient that \( f(q; t) \) satisfy

\[ \frac{df_j}{dt} = \sum p_{jk}(iq)f_k \quad [j, k = 1, \cdots, n], \]

which may be regarded as a system of ordinary DE's because \( q \) is fixed.

Now let \( \Phi \) denote the vector space of all Borel functions \( \phi = \phi(q) \), two such functions being identified when they differ on a set of measure zero. If we write formally

\[ v(x) = \int \phi(q)e^{i_q \cdot x}dQ, \quad dQ = dq_1 \cdots dq_r, \]

then (1) defines, through (3)-(3*), the group

\[ T\phi(q) = \phi(q)e^{iP(iq)}, \quad P(iq) = \|p_{jk}(iq)\|. \]

This is a symbolic solution of (1) in the Fourier transform space \( \Phi \) dual to the space of \( v(x) \).

Further, we can define \( \Phi \) as a Fréchet space, or topological linear space, by making

\[ \phi_n \to \phi \quad \text{mean} \quad \phi_n(q) \to \phi(q) \text{ a.e.} \]

However, this formal solution is unsatisfactory, as a consideration of the case \( u_t = \pm u_{xx} \) shows.

3. Norms on \( \Phi \). Various norms can be defined on \( \Phi \), as follows. Let \( C = C(q) \) be any Borel function from the space \( Q \) of real \( r \)-vectors \( q \), to the space of \( n \times n \) nonsingular complex matrices \( C \). We define

\[ N^2(q; \phi) = (\phi C, \phi C) = \phi C(q)C^*(q)\phi^* \]

as a complex inner product, and correspondingly

\[ N(q; \phi) = [\phi C(q)C^*(q)\phi^*]^{1/2} \geq 0. \]

Since \( C \) is nonsingular, \( N(q; \phi) = 0 \) if and only if \( \phi = 0 \) in \( \Phi \) (i.e., \( \phi_j(q) = 0 \) a.e.).

For any such \( C(q) \), the "norm" \( N(q; \phi) \) is a non-negative real-valued Borel function on \( E_n \times Q \) which, for each real \( r \)-vector \( q \), makes the space of complex \( n \)-vectors \( \phi \) into a unitary space \( E_n(N, q) \).

\footnote{C. Kuratowski, Topologie, first ed., p. 77.}

\footnote{Function "measurable B" in the sense of C. Kuratowski, Topologie, 2d ed, vol. 1, p. 280. We require that the inverse image of a Borel set be a Borel set. Borel functions are measurable, and have the merit of being closed under composition (any Borel function of a Borel function is a Borel function).}
Lemma 1. For any $\phi \in \Phi$, the integral

$$N(\phi) = \int_Q N(q; \phi(q))dQ$$

is defined, as a non-negative real number or $+\infty$.

Proof. First, $N(q; \phi(q))$ is a Borel function for all $\phi \in \Phi$. Since $N(q; \phi) \geq 0$, the conclusion follows immediately. In this proof, it is essential that the functions involved be Borel functions, and not merely measurable.

Lemma 2. Under the norm (8), the set of $\phi \in \Phi$ with $N(\phi) < +\infty$ is a Banach space $B(N)$.

This result is an immediate corollary of standard Lebesgue theory, and the definition of a Banach space. Similar constructions have in fact been used by other authors.$^4$

In the subsequent part of this paper, we shall adopt the following notational conventions. $E_n$ will denote the unitary space of complex $n$-tuples $\phi = (c_1, \ldots, c_n)$ with norm $|\phi| = (\sum_{i=1}^{n-1} c_i \bar{c}_i)^{1/2}$. $E_n(N, q)$ will denote an $r$-parameter family of unitary spaces of complex $n$-tuples $\phi$ with norms $N(q; \phi)$. $B(N)$ will denote the Banach space of complex vector-valued functions $\phi$ with $N(\phi) = \int N(q; \phi)dQ$.

The norm or "modulus" of a linear operator $A(q)$ on $E_n(N, q)$ will be defined by $\|A(q)\|_N = \sup N(q; A(q)[\phi])$ for $N(q; \phi) = 1$. The norm of a bounded linear operator $A$ on $B(N)$ will be defined by $\|A\|_N = \sup N(A[\phi])$ for $N(\phi) = 1$.

Theorem 1. For each $q$, let $T_t(q)$ be a semigroup of linear transformations on the space $E_n$, depending continuously on $q$ and $t \geq 0$. As operators on $E_n(N, q)$, let the $T_t(q)$ have uniformly bounded moduli on $0 < t < 1$. Then the semigroup $\{T_t\}$ on $B(N)$, defined by (5) and (8) for $t > 0$, is a $C_0$-semigroup.

Proof. Because of the continuous dependence of $T_t(q)$ on $q$, $T_t[\phi]$ is always a Borel function; hence $\{T_t\}$ is always a semigroup on $\Phi$. If, for fixed $t$, the $T_t(q)$ have norms bounded by $K(t) < +\infty$ then, substituting (8) into (5), we get

$$N(T_t[\phi]) = K(t) \int_Q N(q; \phi(q))dQ = K(t)\|\phi\|_N.$$
Hence $T_t$ transforms the subspace $B(N)$ of $\Phi$ into itself, and is a linear operator of norm at most $K(t)$ on this Banach space.

It remains [6, p. 18] to show that, for each $\phi \in B(N)$ the orbit $T_t[\phi]$ tends continuously to $\phi$ as $t \downarrow 0$—i.e., that

$$0 = \lim_{t \to 0^+} N(T_t[\phi] - \phi) = \lim_{t \to 0^+} \int_{Q} N(q; T_t[\phi] - \phi) dQ.$$ 

But now, for any $\phi \in B(N)$,

$$0 \leq N(q; T_t[\phi(q)] - \phi(q)) \leq (K(t) + 1)N(q; \phi(q)),$$

where

$$\int_{Q} N(q; \phi(q)) dQ = \|\phi\|_N < + \infty.$$

Hence Lebesgue's Dominated Convergence Theorem\textsuperscript{6} applies, and, since $T_t[\phi(\phi(q))] - \phi(q)) \to 0$ as $t \to 0$ our result is proved.

We note that $T_t$ is a direct integral\textsuperscript{6} of operators on the subspaces of $B(N)$ corresponding to Borel subsets of $Q$; by (8) and (9), the norm of each $T_t$ is the l.u.b. of the norms of the linear operators

$$\{\exp \|P_k(iq)\|\},$$

acting on the $E_n(N; q)$.

4. Regular case. We now assume that the system (1) is regular in the sense of (2), that $\sigma[P] < + \infty$. In this case, for each $q \in Q$, we can choose $C(q)$ so that, under the norm (7'), the modulus of $\{T_t(q)\}$ exceeds $e^{\sigma[P] t}$ by arbitrarily little.

**Lemma 3.** Given $q \in Q$ and $\eta > 0$, we can so choose $C(q)$ that, on $E_n(N; q)$,

$$\|T_t(q)\|_N \leq \exp \{ (\sigma[P(q)] + \eta)t \}, \quad t > 0.$$ 

**Proof.** A slight modification of standard\textsuperscript{7} arguments shows that there always exists a matrix $C(q; \eta)$ for $P(iq)$, such that

$$C^{-1}P(iq)C = J(q; \eta)$$

has entries $\lambda_j(q)$ on the main diagonal, zeros and $\eta$'s just above this diagonal, and all other entries zero. For any such $C(q; \eta)$, the modulus $\|T_t(q)\|_N$ of $T_t(q)$ for the norm (7') is that of $e^{\lambda t}$ operating on the

\textsuperscript{6} For this result, see E. J. McShane, *Integration*, Princeton, 1944, p. 168.


\textsuperscript{7} See for example G. Birkhoff and S. MacLane, *A survey of modern algebra*, rev. ed., Ch. X.
unitary space of complex \( n \)-vectors \( \psi \), under the ordinary norm \( (\psi \psi^*)^{1/2} = |\psi| \). But an explicit calculation shows that this is at most \( \exp \left\{ (\sigma [P(q)] + \eta) t \right\} \).

**Corollary.** Let \( q_0 \in Q \) be given, and let \( \sigma > \sigma(P(q_0)) \). Then \( C = C(q_0) \) exists such that, throughout some neighborhood \( R(q_0) \) of \( q_0 \),

\[
(12) \quad \|T_t(q)\|_N \leq e^{\sigma t} \quad \text{for all } t > 0.
\]

This follows from Lemma 3, because of the continuity of the modulus of \( \|T_t(q)\|_N \), as a function of \( q \), for any fixed \( C_0 = C(q_0) \).

5. **Abstract Cauchy problem.** The preceding results lead directly to a solution of the abstract Cauchy problem, for any regular system (1). We first prove

**Theorem 2.** Let (1) be regular, and let \( \sigma > \sigma(q) \) for all \( q \in Q \). Then a Borel function \( C(q) \) exists, such that (12) holds for all \( q \in Q \).

**Proof.** The \( R(q_0) \) defined by the corollary of Lemma 3 constitute an open covering of \( Q \). But, by Lindelöf's Theorem, one can select from any open covering a countable subcovering \( R_1, R_2, R_3, \ldots \). Replacing each \( R_h \) by \( Q_h = R_h \cap (R_1 \cup \cdots \cup R_{h-1})' \), we get finally a partition of \( Q \) into countably many Borel sets \( Q_h \), such that (12) holds for all \( q \in Q \).

Combining Theorems 1 and 2, we get the following solution to an abstract Cauchy problem corresponding to (1) in a Fourier transform space.

**Corollary.** Let (1) be regular. Then there exists a Borel function \( C(q) \), such that the associated Banach space \( B(N) \) in \( \Phi \) admits a \( C_0 \)-semigroup, whose semi-orbits satisfy (3\*), with \( \|T_t\|_N \leq e^{\sigma t} \) for \( t \geq 0 \).

6. **Differentiable functions.** We now show that, for \( \phi_0(q) \) vanishing fast enough at infinity, the semi-orbits of \( T_t[\phi_0] \) for \( t \geq 0 \) define, through (4), solutions of (1) in the literal sense.

It is well known [2, p. 8] that \( v(x) \) is defined by (4) as a continuous function, whenever \( \phi(q) \in L_1 \). The following result is also easily established.

**Lemma 4.** Let \( \phi \in \Phi \) satisfy

\[
(13) \quad \int |\phi(q)| \left( 1 + \sum |p_{jk}(i q)| \right) dQ < + \infty,
\]

\footnote{See G. T. Whyburn, *Analytic topology*, New York, 1942, p. 4.}

\footnote{This is closely related to some results of [3] and [4]; see also [2, pp. 26, 57, 125] and [7].}
where \( \| \Phi \| = (\Phi \Phi^*)^{1/2} \) is the ordinary norm on \( E_n \). Then, if \( v(x) \) is defined by (4),

\[
\sum_{p} p_{jk}(D)v_k(x) = P_j[v] = \int_{Q} \exp(iq \cdot x) \sum_{p} p_{jk}(iq)\phi_k(q)dQ
\]

exists for all real \( r \)-vectors \( x \).

**Proof.** By (13), the right side of (14), whose integrand is a Borel function and so measurable, is Lebesgue integrable to a continuous function. The rest of the proof is an obvious extension of standard results [2, p. 8].

The following sharper result is less easy.

**Theorem 3.** For all \( \tau, |\tau - t| < \epsilon \), let \( f(q; t) \) satisfy (3*), and also

\[
|f(q; \tau)| (1 + \sum |p_{jk}(iq)|) \leq M(q), \quad \text{a.e.,}
\]

where

\[
\int M(q)dQ < + \infty.
\]

Then (1) is satisfied by the Fourier transform

\[
u(x; t) = \int f(q; t)e^{i\xi \cdot x}dQ.
\]

**Proof.** By Lemma 4, the right side of (1) is defined as a continuous function in space, throughout \( |\tau - t| < \epsilon \). Therefore, if \( |\Delta t| < \epsilon \), the difference quotient \( \Delta u/\Delta t \) is defined. Hence, if (1) failed, we could find an \( x \) and a sequence of \( \Delta t_m \to 0 \), such that

\[
(\Delta u/\Delta t)_m = [u(x; t + \Delta t_m) - u(x; t)]/\Delta t_m
\]

failed to converge to \( P[u(x; t)] \), as defined by (1). We shall now show that this is impossible, which will complete the proof. Indeed, for each \( x \)

\[
\frac{\Delta u}{\Delta t_m} = \frac{1}{\Delta t_m} \int e^{i\xi \cdot x}[f(q; t + \Delta t_m) - f(q; t)]dQ
\]

\[
= \int e^{i\xi \cdot x}dQ \left\{ \frac{1}{\Delta t_m} \int_{t}^{t+\Delta t_m} P[f(q; \tau)]d\tau \right\},
\]

where \( P[f] \) is defined by \( P_j[f] = \sum p_{jk}(iq)f_k \). Hence

\[
\frac{\Delta u}{\Delta t_m} - P[u] = \int_{Q} e^{i\xi \cdot x}dQ \left\{ \frac{1}{\Delta t_m} \int_{t}^{t+\Delta t_m} P[f(q; \tau) - f(q; t)]d\tau \right\}.
\]
in the sense of iterated Lebesgue integration. Since $P$ is a matrix independent of $t$, while $f(q; t)$ depends continuously on $t$, it is clear that, for each fixed $q$,

$$\frac{1}{\Delta t_m} \int_{t}^{t+\Delta t_m} P[f(q; \tau) - f(q; t)]d\tau = \Delta_m(q) \to 0 \text{ as } \Delta t_m \to 0.$$ 

Hence, by Lebesgue’s Dominated Convergence Theorem and (16), $\Delta u/\Delta t_m \to P[u]$ for each fixed $x$ provided

(17) \[ |\Delta_m(q)| \leq K(q), \quad \int_{Q} K(q)dQ < + \infty. \]

But, by definition,

$$|\Delta_m \Delta_m|^2 = \sum_{j=1}^{n} \left( \int_{t}^{t+\Delta t_m} \sum_{k=1}^{n} p_{jk}(i \omega) [f_h(q, \tau) - f_h(q; t)]d\tau \right)^2.$$ 

Applying Schwarz’ Inequality, we get

$$|\Delta_m \Delta_m| \leq \sum_{j=1}^{m} \left( \int_{t}^{t+\Delta t_m} \left( \sum_{k=1}^{n} p_{jk} \right)^{1/2} \left( |f(\tau)| + |f(t)| \right)d\tau \right)^2$$

$$\leq 2M(q)\Delta t_m \text{ by (13*).}$$

This gives (17), with $K(q) = 2M(q)$.

7. Concrete Cauchy problem. Suitably combining Theorem 3 with the methods used in proving Theorems 1–2, we can also solve the concrete Cauchy problem. First, we observe that, in §4, the results are unchanged if we replace $C(q)$ by $w(q)C(q)$ and $C(q_0)$ by $w(q_0)C(q_0)$, for any positive scalars $w(q)$ and $w(q_0)$. In particular, the modulus of the $T(q)$ is unchanged, while $N(q; \phi)$ is replaced by $N_w(q; \phi) = w(q)N(q; \phi)$.

By choosing the $w(q)$ sufficiently large, however, we can make semi-orbits in the complex Banach space $B(N_w)$ of the $\phi(q) \in \Phi$ such that (cf. (8))

$$\int N(q; \phi(q))w(q)dQ < + \infty,$$

consist entirely of functions which satisfy the conditions of §6.

More precisely, any Borel function $C(q)$ of the Corollary to Theorem 2 can be modified in the following way. For $q \in Q$, $q$ is in some $Q_h$ and $C(q) = C(q_h)$ where $Q_h \subseteq R_h = R(q_h)$. Let $w(q) = w(q_h)$ where $w(q_h) > 0$ is so chosen that in $R(q_h)$
Then \( w(q) \) is a Borel function since it is constant on each Borel set \( Q_h \), and \( w(q) C(q) \) defines a Banach space \( B(N_w) \) as before. The modulus of the semigroup (5) is given by

\[
\| T_t(q) \|_{N_w} \leq e^{\epsilon t},
\]

where

\[
N_w(q; \phi) = w(q) N(q; \phi).
\]

It will follow that, on any interval \( |t - \tau| < \epsilon \) of any semi-orbit in \( B(N_w) \), the hypotheses of Theorem 3 will be satisfied, with \( M(q) = \sup_e e^{\epsilon \tau} N_w(q; \phi(q)) \). This proves

**Theorem 4.** For suitable \( B(N) \), every semi-orbit in the corollary of Theorem 2 represents an actual solution of (1).

We now define the system (1) to be hyperbolic if and only if \( P \) and \(-P\) are both regular (see [1, §9] for a discussion of alternative definitions). Since the substitution of \(-P\) for \( P \) leaves the modified Jordan canonical form of §4 unchanged except for sign, (10) holds with the substitution of \( |t| \) for \( t \). This proves the

**Corollary.** If (1) is hyperbolic, then the \( C_0 \)-semigroups of Theorem 4 and Corollary of Theorem 2 are parts of \( C_0 \)-groups, with moduli \( \| T_t \| \leq e^{\epsilon |t|} \).

**Bibliography**


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