ON RIESZ SUMMABILITY OF FOURIER SERIES

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1. Let \( \phi(t) \) be an even function, integrable in Lebesgue sense and periodic with period \( 2\pi \). Let

\[
\phi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt.
\]

Let \( \lambda(x) \) be a continuous function, monotone increasing in \((0, \infty)\) and tends to infinity as \( x \to \infty \). The series \( \sum a_n \) is said to be summable \((R, \lambda(\omega), r)\) if \( c_r(\omega)/(\lambda(\omega))^r \to s \) as \( \omega \to \infty \), where

\[
c_r(\omega) = \frac{1}{2} (\lambda(\omega) - \lambda(0))^ra_0 + \sum_{n \leq \omega} \frac{\lambda(n)}{\lambda(n)} a_n.
\]

Throughout this note, we assume \( s = 0 \), and on writing \( \lambda(x) = e^{\mu(x)} \), \( \mu(x) \) satisfies the following conditions (A):

(i) \( \mu(x) \) is differentiable and monotone increasing for \( x > A \) and \( \mu(x) > 1 \) as \( x \to \infty \),

(ii) \( \mu'(x) \) is monotone decreasing for \( x > A \) and \( \mu'(x) \to 0 \) as \( x \to \infty \),

(iii) \( (\mu'(x))^2 \geq -2\mu''(x) \geq 0 \), \( (\mu'(x))^3 \geq -(3\mu'(x)\mu''(x) + \mu'''(x)) \) for \( x > A \),

(iv) \[
U(x) = \int_{x}^{\infty} \frac{du}{u\mu(u)} \text{ exists for } x > A
\]

and

(a) \[
x(\mu'(x))^\Delta U \left( \frac{1}{\mu'(x)} \right) = O(1)
\]

for some \( 0 < \Delta < 2 \),

(b) \( x^\Delta U(x) \) being monotone increasing for some \( 0 < \Delta < 2 - \Delta \), \( \mu(x) \) may be taken in the following forms:

(1) \( (\log x)^{1+\Delta} \) \( (\Delta > 0, x > A > 1) \),

(2) \( x^\alpha \) \( (0 < \alpha < 1/2, x > A > 0) \),

(3) \( x^\alpha (\log x)^{-\Delta} \) \( (\Delta > 0, 0 < \alpha < 1/2, x > A > 1) \).

Hardy and Littlewood have derived a convergence test for the oscillating series \( \sum a_n [2; 3; 4] \) which contains two conditions: one is

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concerning the behavior of \( \phi(t) \) near the point \( t = 0 \), e.g., the continuity condition. The other is concerning the magnitude of \( a_n \), e.g., the coefficients condition. Morgan [8] developed Hardy-Littlewood's result so as to permit varying types of conditions on \( \phi(t) \) and \( a_n \). Wang [5; 10; 11] proved Hardy-Littlewood's test in one-sided Tauberian condition for \( a_n \). Sunouchi [9] joined the results of Morgan and Wang and established another type of convergence test for the series \( \sum a_n \). Jurkat [6] developed Sunouchi's result as same as Hardy-Littlewood's to further steps. Very recently, Kanno [7] has extended Wang's theorem to the fractional integral \( \phi_{\alpha}(t) (\alpha \geq 1) \) of \( \phi(t) \). In the present note, we give the above investigations an exhaustive solution. Our result includes, however, to some degrees, the works of the previous authors. It is known that [10], for \( r > 0, \omega > 0 \),

\[
c_r(\omega) = \frac{2}{\pi} \int_0^\pi \phi(t) E_r(\omega, t) dt + o((\lambda(\omega))^r),
\]

where

\[
E_r(\omega, t) = \frac{r}{t} \int_0^\omega (\lambda(\omega) - \lambda(x))^{r-1} \lambda'(x) \sin xtdx.
\]

In addition, we define:

\[
\lambda(x) = \begin{cases} e^{\mu(x)} & (x > A), \\ \lambda(A) + \lambda'(A)(x - A) & (0 \leq x \leq A). \end{cases}
\]

2. The following lemmas are required.

Lemma 1. For \( \omega > A \),

\[
E_2(\omega, t) = O(e^{2\mu(\omega)} \omega)
\]

uniformly in \( t \),

\[
E_2(\omega, t) = O(e^{\mu(\omega)} t^{-2}) + O(e^{2\mu(\omega)} \mu'(\omega) t^{-2}),
\]

\[
\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{2\mu(\omega)} \omega t^{-1}),
\]

and

\[
\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{\mu(\omega)} t^{-3}) + O(e^{2\mu(\omega)} (\mu'(\omega))^2 \omega t^{-3}).
\]

Proof. By the second mean value theorem,\(^1\)

\(^1\) B is a numerical constant not necessarily the same at each occurrence.
\[ E_2(\omega, t) = \int_0^\omega (\lambda(\omega) - \lambda(x))^2 \cos x \, dx \]
\[ = (\lambda(\omega) - B)^2 \int_0^\omega \cos x \, dx \]
\[ = O(e^{2\mu(\omega)}). \]

\[ \frac{\partial}{\partial t} E_2(\omega, t) = - \int_0^\omega (\lambda(\omega) - \lambda(x))^2 x \sin x \, dx \]
\[ = - (\lambda(\omega) - B)^2 \omega \int_0^\omega \sin x \, dx \]
\[ = O(e^{2\mu(\omega)}). \]

for \( \omega > A \), by the second mean value theorem, where \( 0 < \omega' < \omega'' < \omega \).

Now,

\[ E_2(\omega, t) = \frac{2}{t} \left( \int_0^A + \int_A^\omega \right) (\lambda(\omega) - \lambda(x)) \lambda'(x) \sin x \, dx \]
\[ = L_1 + L_2. \]

\[ L_1 = \frac{2}{t} (\lambda(\omega) - B) \lambda'(A) \int_0^\omega \sin x \, dx = O(e^{\mu(\omega)} t^{-2}), \]

\[ L_2 = \frac{2}{t} (\lambda(\omega) - B) \lambda'(\omega) \int_0^\omega \sin x \, dx, \]

since, by (iii) of the conditions (A), \( \lambda'(x) \) is monotone increasing in \( (A, \omega) \). It follows that

\[ L_2 = O(e^{2\mu(\omega)} \mu'(\omega) t^{-2}). \]

Thus,

\[ E_2(\omega, t) = O(e^{\mu(\omega)} t^{-2}) + O(e^{2\mu(\omega)} \mu'(\omega) t^{-2}) \]

for \( \omega > A \). Further,

\[ \frac{\partial L_1}{\partial t} = - \frac{2\lambda'(A)}{t^2} \int_0^A (\lambda(\omega) - \lambda(x)) \sin x \, dx \]
\[ + \frac{2\lambda'(A)}{t} \int_0^A (\lambda(\omega) - \lambda(x)) x \cos x \, dx \]
\[ = O(e^{\mu(\omega)} t^{-3}). \]

By integration by parts once, we find
\[ L_2 = \frac{2}{t^2} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At + \frac{2}{t^2} \int_A^\omega \frac{d}{dx} ((\lambda(\omega) - \lambda(x))\lambda'(x) \cos xtdx = \frac{2}{t^2} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At \]

\[ - \frac{2}{t^2} \int_A^\omega ((\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x)) \cos xtdx. \]

Denote \( H(\omega, x) = (\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x). \) Then

\[ \frac{\partial L_2}{\partial t} = \frac{4}{t^3} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At + \frac{2A}{t^3} (\lambda(\omega) - \lambda(A)) \sin At + \frac{4}{t^3} \int_A^\omega H(\omega, x) \cos xtdx + \frac{2}{t^2} \int_A^\omega H(\omega, x)x \sin xtdx \]

\[ = O(\omega^2 t^{-3}) + (I_1 + I_2 + I_3 + I_4). \]

\[ I_1 = \frac{4}{t^3} \int_A^\omega (\lambda'(x))^2 \cos xtdx = \frac{4}{t^3} (\lambda'(\omega))^2 \int_{\omega'} \cos xtdx = O((\lambda'(\omega))^2 \omega t^{-3}). \]

\[ I_2 = -\frac{4}{t^3} \int_A^\omega (\lambda(\omega) - \lambda(x))\lambda''(x) \cos xtdx \]

\[ = -\frac{4}{t^3} (\lambda(\omega) - B)\lambda''(\omega) \int_{\omega'} \cos xtdx, \]

since, by (iii) of the conditions \((A), \lambda''(x)\) is monotone increasing in \((A, \omega). \) It follows that

\[ I_2 = O(t^{-3}(\lambda(\omega) - B)e^{\theta(\omega)}((\mu'(\omega))^2 + \mu''(\omega)) \omega) \]

\[ = O(e^{2\theta(\omega)}(\mu'(\omega))^2 \omega t^{-3}) + O(e^{2\theta(\omega)} | \mu''(\omega) | \omega t^{-3}). \]

In view of \((\mu'(x))^2 \geq -2\mu''(x) \geq 0\) for \( x > A, \) the second term can be absorbed in the first, so that

\[ I_2 = O(e^{2\theta(\omega)}(\mu'(\omega))^2 \omega t^{-3}). \]

\[ I_2 = \frac{2}{t^2} \int_A^\omega (\lambda'(x))^2 x \sin xtdx \]

\[ = \frac{2}{t^2} (\lambda'(\omega))^2 \omega \int_{\omega'} \sin xtdx = O((\lambda'(\omega))^2 \omega t^{-3}). \]
\[ I_4 = -\frac{2}{t^2} \int_{A}^{\omega} (\lambda(\omega) - \lambda(x))\lambda''(x)x \sin x \, dx \]

\[ = -\frac{2}{t^2} (\lambda(\omega) - B)\lambda''(\omega) \omega \int_{\omega}^{\omega''} \sin x \, dx \]

\[ = -\frac{2}{t^2} (\lambda(\omega) - B)e^{\mu(\omega)}((\mu'(\omega))^2 + \mu''(\omega)) \omega \int_{\omega}^{\omega''} \sin x \, dx \]

\[ = O(e^{2\mu(\omega)}(\mu'(\omega))^2 \omega t^{-3}) + O(e^{2\mu(\omega)} | \mu''(\omega) | \omega t^{-3}) \]

\[ = O(e^{2\mu(\omega)}(\mu'(\omega))^2 \omega t^{-3}). \]

Thus, \((\partial/\partial t)E_2(\omega, t) = O(e^{\mu(\omega)}t^{-3}) + O(e^{2\mu(\omega)}(\mu'(\omega))^2 \omega t^{-3})\) for \(\omega > A\). This proves the lemma.

**Lemma 2.** If \(\sum a_n\) is summable \((R, \lambda(\omega), r)\) and

\[ a_n > -K \frac{\lambda(n) - \lambda(n - 1)}{\lambda(n)}, \]

then \(\sum a_n\) converges.

This lemma is known. [1].

3. Now, we are in a position to establish the following

**Theorem 1.** If

\[ \phi_1(t) = \int_0^t \phi(u) \, du = \mathcal{O}(1) \left( \frac{1}{1 - t} \right) \quad (t \to 0), \]

then the series \(\sum a_n\) is summable \((R, \lambda(\omega), 2)\) to zero.

**Proof.** We have

\[ \frac{\pi}{2} c_2(\omega) = \int_0^\pi \phi(t) E_2(\omega, t) \, dt + o(e^{2\mu(\omega)}) \]

\[ = \left( \int_0^{1/A-\epsilon} + \int_{1/A-\epsilon}^\pi \right) \phi(t) E_2(\omega, t) \, dt + o(e^{2\mu(\omega)}). \]

By the second relation of Lemma 1,

\[ \int_{1/A-\epsilon}^\pi \phi(t) E_2(\omega, t) \, dt \]

\[ = O\left( e^{\mu(\omega)} \int_{1/A-\epsilon}^\pi \left| \frac{\phi(t)}{t^2} \right| \, dt \right) + O\left( e^{2\mu(\omega)} \mu'(\omega) \int_{1/A-\epsilon}^\pi \left| \frac{\phi(t)}{t^2} \right| \, dt \right) \]

\[ = O(e^{\mu(\omega)}) + O(e^{2\mu(\omega)} \mu'(\omega)) = o(e^{2\mu(\omega)}), \]
since $\mu'(x) \to 0$ as $x \to A$. By integration by parts,

$$\int_0^{1/A-\epsilon} \phi(t) E_2(\omega, t) dt = (\phi_1(t) E_2(\omega, t))_0^{1/A-\epsilon} - \int_0^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt.$$

The integrated term is easily seen to be $o(e^{2\mu(\omega)})$ by the first and the second relations of Lemma 1. Write

$$\int_0^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt = \left( \int_0^{\mu'(\omega)} + \int_{\mu'(\omega)}^{1/A-\epsilon} \right) \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt = J_1 + J_2.$$

$$J_1 = o \left( e^{2\mu(\omega)} \omega \int_0^{\mu'(\omega)} t^{\Delta-1} U \left( \frac{1}{t} \right) dt \right)$$

$$= o \left( e^{2\mu(\omega)} \omega (\mu'(\omega))^\Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)$$

$$= o(e^{2\mu(\omega)})$$

by (iv) of the conditions (A).

$$J_2 = \int_{\mu'(\omega)}^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt$$

$$= o \left( e^{\mu(\omega)} \int_{\mu'(\omega)}^{1/A-\epsilon} t^{\Delta-3} U(t) dt \right) + o \left( e^{2\mu(\omega)} (\mu'(\omega))^2 \omega \int_{\mu'(\omega)}^{1/A-\epsilon} t^{\Delta-3} U(t) dt \right)$$

$$= o \left( e^{\mu(\omega)} (\mu'(\omega))^{-\Delta} U \left( \frac{1}{\mu'(\omega)} \right) \int_{\mu'(\omega)}^r t^{\Delta-3} dt \right)$$

$$+ o \left( e^{2\mu(\omega)} (\mu'(\omega))^{2-\Delta} \omega U \left( \frac{1}{\mu'(\omega)} \right) \int_{\mu'(\omega)}^r t^{\Delta-3} dt \right)$$

$$= o \left( e^{\mu(\omega)} (\mu'(\omega))^\Delta U \left( 1 \left/ \mu'(\omega) \right. \right) \right) + o \left( e^{2\mu(\omega)} (\mu'(\omega))^\Delta U \left( 1 \left/ \mu'(\omega) \right. \right) \right) + o(e^{2\mu(\omega)})$$

since $e^{\mu(x)} (\mu'(x))^2$ is positive and monotone increasing for $x > A$ by (iii) of the conditions (A). This proves the theorem.

By means of Theorem 1 and Lemma 2, we obtain immediately

**Theorem 2.** If

\[(i) \quad \phi_1(t) = o \left( t^\Delta U \left( 1 \left/ t \right. \right) \right) \quad (t \to 0),\]
\[ a_n > - K \frac{\lambda(n) - \lambda(n - 1)}{\lambda(n)} \]

for some \( K > 0 \), then \( \sum a_n \) converges to zero.

4. Last, we illustrate by some special cases derived from our result:

(a) Let \( 0 < \alpha < 1/2 \). If

\[ (i) \quad \phi_1(t) = o \left( t^{1/\alpha}(\log \frac{1}{t})\Delta \right) \quad (t \to +0), \]

\[ (ii) \quad a_n > - Kn^{\alpha-1} (\log n)^{-\Delta} \]

for some \( K > 0 \), then \( \sum a_n \) converges to zero.

(b) \( \alpha < 0 \). If

\[ (i) \quad \phi_1(t) = o(t^{1/(1-\alpha)}) \quad (t \to +0), \]

\[ (ii) \quad a_n > - Kn^{\alpha-1} \]

then \( \sum a_n \) converges to zero.

(c) If

\[ (i) \quad \phi_1(t) = o \left( t \left( \log \frac{1}{t} \right)^{-\alpha} \right) \quad (t \to +0), \]

\[ (ii) \quad a_n > Kn^{-1} (\log n)^{\Delta} \]

for some \( \Delta > 0 \), then \( \sum a_n \) converges to zero.

REFERENCES


ON THE IDENTITY OF FUNCTION SPACES ON CARTESIAN PRODUCT SPACES

JOHN C. HOLLADAY

For \( i = 1, \ldots, n \), let \( S_i \) be a compact Hausdorff space and \( F_i \) be a closed linear subspace of the complex Banach space \( C(S_i) \), the set of all continuous functions from \( S_i \) to the complex numbers. Let \( S_1 \times \cdots \times S_n \) be the Cartesian Product of \( S_1, \ldots, S_n \).

Define \( F_1 \ast \cdots \ast F_n \) as \( \{ \phi | \phi \in C(S_1 \times \cdots \times S_n); \text{ for any } i = 1, \ldots, n \text{ and } (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \phi(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in F_i \} \). Also, define \( F_1 \otimes \cdots \otimes F_n \) as the closure of the space of linear combinations of functions of the form \( \phi(s_1, \ldots, s_n) = f_1(s_1) \times \cdots \times f_n(s_n) \) where each \( f_i \in F_i \), where we base the topology on the norm, \( \| \phi \| = \max_{s_1, \ldots, s_n} |\phi(s_1, \ldots, s_n)| \).

It is easily shown that \( F_1 \otimes \cdots \otimes F_n \) is a subspace of \( F_1 \ast \cdots \ast F_n \) and that if \( F_2 \) is one-dimensional, then \( F_1 \otimes F_2 = F_1 \ast F_2 \). Furthermore, by using continuous partitions of unity, one may show that \( F_1 \otimes C(S_2) = F_1 \ast C(S_2) \). Therefore, if all but at most one of the \( F_i \) are either one-dimensional or \( C(S_i) \), then \( F_1 \otimes \cdots \otimes F_n = F_1 \ast \cdots \ast F_n \). However, it is not known whether for all cases \( F_1 \otimes \cdots \otimes F_n \) will equal \( F_1 \ast \cdots \ast F_n \) or not. Although this question is not fully answered here, the purpose of this paper is to give a partial answer to this question. The results and arguments of this paper also apply to real-valued function spaces.

1. Lemma. Let \( F \) be a closed linear subspace of \( C(S_1) \) and \( G \) a closed linear subspace of \( C(S_2) \). Let \( H \) be a closed subspace of \( G \) differing from \( G \) by only one dimension. Then \( F \otimes G = F \ast G \) implies that \( F \otimes H = F \ast H \).

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1 A small part of the work done under an AEC Predoctoral Fellowship at Yale University, year 1952-1953, under the kind and patient guidance of Dr. Charles E. Rickart.

2 Under Proposition 37 of his first paper of the Amer. Math. Soc. Memoirs, no. 16, Alexandre Grothendieck discusses a number of conjectures which are equivalent to this one.

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