ON RIESZ SUMMABILITY OF FOURIER SERIES

FU CHENG HSIANG

1. Let $\phi(t)$ be an even function, integrable in Lebesgue sense and periodic with period 2π. Let

$$\phi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt.$$

Let $\lambda(x)$ be a continuous function, monotone increasing in $(0, \infty)$ and tends to infinity as $x \to \infty$. The series $\sum a_n$ is said to be summable $(R, \lambda(\omega), r)$ if $c_r(\omega)/(\lambda(\omega))^r \to s$ as $\omega \to \infty$, where

$$c_r(\omega) = \frac{1}{2} (\lambda(\omega) - \lambda(0))^r a_0 + \sum_{n \leq \omega} (\lambda(\omega) - \lambda(n))^r a_n.$$

Throughout this note, we assume $s=0$, and on writing $\lambda(x) = e^{\mu(x)}$, $\mu(x)$ satisfies the following conditions (A):

(i) $\mu(x)$ is differentiable and monotone increasing for $x > A$ and $\mu(x) > 1$ as $x \to \infty$,

(ii) $\mu'(x)$ is monotone decreasing for $x > A$ and $\mu'(x) \to 0$ as $x \to \infty$,

(iii) $(\mu'(x))^2 \geqq -2\mu''(x) \geqq 0$, $(\mu'(x))^3 \geqq -(3\mu'(x)\mu''(x) + \mu'''(x))$ for $x > A$,

(iv) $$U(x) = \int_x^{\infty} \frac{du}{u\mu(u)} \text{ exists for } x > A$$

and

(a) $$x(\mu'(x))^{\Delta} U\left(\frac{1}{\mu'(x)}\right) = O(1)$$

for some $0 < \Delta < 2$,

(b) $x^{\Delta} U(x)$ being monotone increasing for some $0 < \Lambda < 2 - \Delta$, $\mu(x)$ may be taken in the following forms:

(1) $(\log x)^{1+\Delta}$ $(\Delta > 0, x > A > 1)$, (2) x^{α} $(0 < \alpha < 1/2, x > A > 0)$,

(3) $x^{\alpha} (\log x)^{-\Delta}$ $(\Delta > 0, 0 < \alpha < 1/2, x > A > 1)$.

Hardy and Littlewood have derived a convergence test for the oscillating series $\sum a_n$ [**2**; **3**; **4**] which contains two conditions: one is

Received by the editors April 8, 1957 and, in revised form, June 21, 1957.

concerning the behavior of $\phi(t)$ near the point $t=0$, e.g., the continuity condition. The other is concerning the magnitude of a_n, e.g., the coefficients condition. Morgan [**8**] developed Hardy-Littlewood's result so as to permit varying types of conditions on $\phi(t)$ and a_n. Wang [**5**; **10**; **11**] proved Hardy-Littlewood's test in one-sided Tauberian condition for a_n. Sunouchi [**9**] joined the results of Morgan and Wang and established another type of convergence test for the series $\sum a_n$. Jurkat [**6**] developed Sunouchi's result as same as Hardy-Littlewood's to further steps. Very recently, Kanno [**7**] has extended Wang's theorem to the fractional integral $\phi_\alpha(t)$ $(\alpha \geqq 1)$ of $\phi(t)$. In the present note, we give the above investigations an exhaustive solution. Our result includes, however, to some degrees, the works of the previous authors. It is known that [**10**], for $r>0$, $\omega>0$,

$$c_r(\omega) = \frac{2}{\pi}\int_0^\pi \phi(t)E_r(\omega, t)dt + o((\lambda(\omega))^r),$$

where

$$E_r(\omega, t) = \frac{r}{t}\int_0^\omega (\lambda(\omega) - \lambda(x))^{r-1}\lambda'(x) \sin xt dx.$$

In addition, we define:

$$\lambda(x) = \begin{cases} e^{\mu(x)} & (x > A), \\ \lambda(A) + \lambda'(A)(x - A) & (0 \leqq x \leqq A). \end{cases}$$

2. The following lemmas are required.

LEMMA 1. *For* $\omega > A$,

$$E_2(\omega, t) = O(e^{2\mu(\omega)}\omega)$$

uniformly in t,

$$E_2(\omega, t) = O(e^{\mu(\omega)}t^{-2}) + O(e^{2\mu(\omega)}\mu'(\omega)t^{-2}),$$

$$\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{2\mu(\omega)}\omega t^{-1}),$$

and

$$\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{\mu(\omega)}t^{-3}) + O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}).$$

PROOF. By the second mean value theorem,[1]

[1] B is a numerical constant not necessarily the same at each occurrence.

$$E_2(\omega, t) = \int_0^\omega (\lambda(\omega) - \lambda(x))^2 \cos xt dx$$

$$= (\lambda(\omega) - B)^2 \int_0^{\omega'} \cos xt dx$$

$$= O(e^{2\mu(\omega)}\omega).$$

$$\frac{\partial}{\partial t} E_2(\omega, t) = -\int_0^\omega (\lambda(\omega) - \lambda(x))^2 x \sin xt dx$$

$$= -(\lambda(\omega) - B)^2\omega \int_{\omega'}^{\omega''} \sin xt dx$$

$$= O(e^{2\mu(\omega)}\omega t^{-1})$$

for $\omega > A$, by the second mean value theorem, where $0 < \omega' < \omega'' < \omega$.

Now,

$$E_2(\omega, t) = \frac{2}{t}\left(\int_0^A + \int_A^\omega\right)(\lambda(\omega) - \lambda(x))\lambda'(x) \sin xt dx$$

$$= L_1 + L_2.$$

$$L_1 = \frac{2}{t}(\lambda(\omega) - B)\lambda'(A) \int_0^\tau \sin xt dx = O(e^{\mu(\omega)}t^{-2}),$$

$$L_2 = \frac{2}{t}(\lambda(\omega) - B)\lambda'(\omega) \int_{\omega'}^{\omega''} \sin xt dx,$$

since, by (iii) of the conditions (A), $\lambda'(x)$ is monotone increasing in (A, ω). It follows that

$$L_2 = O(e^{2\mu(\omega)}\mu'(\omega)t^{-2}).$$

Thus,

$$E_2(\omega, t) = O(e^{\mu(\omega)}t^{-2}) + O(e^{2\mu(\omega)}\mu'(\omega)t^{-2})$$

for $\omega > A$. Further,

$$\frac{\partial L_1}{\partial t} = -\frac{2\lambda'(A)}{t^2}\int_0^A (\lambda(\omega) - \lambda(x)) \sin xt dx$$

$$+ \frac{2\lambda'(A)}{t}\int_0^A (\lambda(\omega) - \lambda(x))x \cos xt dx$$

$$= O(e^{\mu(\omega)}t^{-3}).$$

By integration by parts once, we find

$$L_2 = \frac{2}{t^2}(\lambda(\omega) - \lambda(A))\lambda'(A)\cos At$$
$$+ \frac{2}{t^2}\int_A^\omega \frac{d}{dx}((\lambda(\omega) - \lambda(x))\lambda'(x)\cos xtdx$$
$$= \frac{2}{t^2}(\lambda(\omega) - \lambda(A))\lambda'(A)\cos At$$
$$- \frac{2}{t^2}\int_A^\omega ((\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x))\cos xtdx.$$

Denote $H(\omega, x) = (\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x)$. Then

$$\frac{\partial L_2}{\partial t} = \frac{4}{t^3}(\lambda(\omega) - \lambda(A))\lambda'(A)\cos At + \frac{2A}{t^2}(\lambda(\omega) - \lambda(A))\sin At$$
$$+ \frac{4}{t^3}\int_A^\omega H(\omega, x)\cos xtdx + \frac{2}{t^2}\int_A^\omega H(\omega, x)x\sin xtdx$$
$$= O(e^{\mu(\omega)}t^{-3}) + (I_1 + I_2 + I_3 + I_4).$$
$$I_1 = \frac{4}{t^3}\int_A^\omega (\lambda'(x))^2\cos xtdx = \frac{4}{t^3}(\lambda'(\omega))^2\int_{\omega'}^\omega \cos xtdx$$
$$= O((\lambda'(\omega))^2\omega t^{-3}).$$
$$I_2 = -\frac{4}{t^3}\int_A^\omega (\lambda(\omega) - \lambda(x))\lambda''(x)\cos xtdx$$
$$= -\frac{4}{t^3}(\lambda(\omega) - B)\lambda''(\omega)\int_{\omega'}^{\omega''} \cos xtdx,$$

since, by (iii) of the conditions (A), $\lambda''(x)$ is monotone increasing in (A, ω). It follows that

$$I_2 = O(t^{-3}(\lambda(\omega) - B)e^{\mu(\omega)}((\mu'(\omega))^2 + \mu''(\omega))\omega)$$
$$= O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}) + O(e^{2\mu(\omega)}\,|\,\mu''(\omega)\,|\,\omega t^{-3}).$$

In view of $(\mu'(x))^2 \geqq -2\mu''(x) \geqq 0$ for $x > A$, the second term can be absorbed in the first, so that

$$I_2 = O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}).$$
$$I_2 = \frac{2}{t^2}\int_A^\omega (\lambda'(x))^2 x\sin xtdx$$
$$= \frac{2}{t^2}(\lambda'(\omega))^2\omega\int_{\omega'}^\omega \sin xtdx = O((\lambda'(\omega))^2\omega t^{-3}).$$

$$
\begin{aligned}
I_4 &= -\frac{2}{t^2}\int_A^{\omega}(\lambda(\omega)-\lambda(x))\lambda''(x)x\sin xt dx \\
&= -\frac{2}{t^2}(\lambda(\omega)-B)\lambda''(\omega)\omega\int_{\omega'}^{\omega''}\sin xt dx \\
&= -\frac{2}{t^2}(\lambda(\omega)-B)e^{\mu(\omega)}((\mu'(\omega))^2+\mu''(\omega))\omega\int_{\omega'}^{\omega''}\sin xt dx \\
&= O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3})+O(e^{2\mu(\omega)}\mid\mu''(\omega)\mid\omega t^{-3}) \\
&= O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}).
\end{aligned}
$$

Thus, $(\partial/\partial t)E_2(\omega, t)=O(e^{\mu(\omega)}t^{-3})+O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3})$ for $\omega>A$. This proves the lemma.

LEMMA 2. *If* $\sum a_n$ *is summable* $(R, \lambda(\omega), r)$ *and*

$$a_n > -K\frac{\lambda(n)-\lambda(n-1)}{\lambda(n)},$$

then $\sum a_n$ *converges.*

This lemma is known. [1].

3. Now, we are in a position to establish the following

THEOREM 1. *If*

$$\phi_1(t)=\int_0^t\phi(u)du=o\left(t^{\Delta}U\left(\frac{1}{t}\right)\right) \qquad (t\to+0),$$

then the series $\sum a_n$ *is summable* $(R, \lambda(\omega), 2)$ *to zero.*

PROOF. We have

$$
\begin{aligned}
\frac{\pi}{2}c_2(\omega) &= \int_0^{\pi}\phi(t)E_2(\omega, t)dt+o(e^{2\mu(\omega)}) \\
&= \left(\int_0^{1/A-\epsilon}+\int_{1/A-\epsilon}^{\pi}\right)\phi(t)E_2(\omega, t)dt+o(e^{2\mu(\omega)}).
\end{aligned}
$$

By the second relation of Lemma 1,

$$
\begin{aligned}
&\int_{1/A-\epsilon}^{\pi}\phi(t)E_2(\omega, t)dt \\
&= O\left(e^{\mu(\omega)}\int_{1/A-\epsilon}^{\pi}\frac{\mid\phi(t)\mid}{t^2}dt\right)+O\left(e^{2\mu(\omega)}\mu'(\omega)\int_{1/A-\epsilon}^{\pi}\frac{\mid\phi(t)\mid}{t^2}dt\right) \\
&= O(e^{\mu(\omega)})+O(e^{2\mu(\omega)}\mu'(\omega)) = o(e^{2\mu(\omega)}),
\end{aligned}
$$

since $\mu'(\omega)\to 0$ as $\omega\to\infty$. By integration by parts,

$$\int_0^{1/A-\epsilon}\phi(t)E_2(\omega,t)dt = (\phi_1(t)E_2(\omega,t))_0^{1/A-\epsilon} - \int_0^{1/A-\epsilon}\phi_1(t)\frac{\partial}{\partial t}E_2(\omega,t)dt.$$

The integrated term is easily seen to be $o(e^{2\mu(\omega)})$ by the first and the second relations of Lemma 1. Write

$$\begin{aligned}\int_0^{1/A-\epsilon}\phi_1(t)\frac{\partial}{\partial t}E_2(\omega,t)dt &= \left(\int_0^{\mu'(\omega)} + \int_{\mu'(\omega)}^{1/A-\epsilon}\right)\phi_1(t)\frac{\partial}{\partial t}E_2(\omega,t)dt\\ &= J_1 + J_2.\end{aligned}$$

$$\begin{aligned}J_1 &= o\left(e^{2\mu(\omega)}\omega\int_0^{\mu'(\omega)} t^{\Delta-1}U\left(\frac{1}{t}\right)dt\right)\\ &= o\left(e^{2\mu(\omega)}\omega(\mu'(\omega))^{\Delta}U\left(\frac{1}{\mu'(\omega)}\right)\right)\\ &= o(e^{2\mu(\omega)})\end{aligned}$$

by (iv) of the conditions (A).

$$\begin{aligned}J_2 &= \int_{\mu'(\omega)}^{1/A-\epsilon}\phi_1(t)\frac{\partial}{\partial t}E_2(\omega,t)dt\\ &= o\left(e^{\mu(\omega)}\int_{\mu'(\omega)}^{1/A-\epsilon} t^{\Delta-3}U(t)dt\right) + o\left(e^{2\mu(\omega)}(\mu'(\omega))^2\omega\int_{\mu'(\omega)}^{1/A-\epsilon} t^{\Delta-3}U(t)dt\right)\\ &= o\left(e^{\mu(\omega)}(\mu'(\omega))^{-\Lambda}U\left(\frac{1}{\mu'(\omega)}\right)\int_{\mu'(\omega)}^{\tau} t^{\Delta+\Lambda-3}dt\right)\\ &\quad + o\left(e^{2\mu(\omega)}(\mu'(\omega))^{2-\Lambda}\omega U\left(\frac{1}{\mu'(\omega)}\right)\int_{\mu'(\omega)}^{\tau} t^{\Delta+\Lambda-3}dt\right)\\ &= o\left(e^{\mu(\omega)}(\mu'(\omega))^{\Delta-2}U\left(\frac{1}{\mu'(\omega)}\right)\right) + o\left(e^{2\mu(\omega)}(\mu'(\omega))^{\Delta}\omega U\left(\frac{1}{\mu'(\omega)}\right)\right)\\ &= o\left(e^{2\mu(\omega)}(\mu(\omega)(\mu'(\omega))^2)^{-1}(\mu'(\omega))^{\Delta}U\left(\frac{1}{\mu'(\omega)}\right)\right) + o(e^{2\mu(\omega)})\\ &= o(e^{2\mu(\omega)})\end{aligned}$$

since $e^{\mu(x)}(\mu'(x))^2$ is positive and monotone increasing for $x>A$ by (iii) of the conditions (A). This proves the theorem.

By means of Theorem 1 and Lemma 2, we obtain immediately

THEOREM 2. *If*

(i) $$\phi_1(t) = o\left(t^{\Delta}U\left(\frac{1}{t}\right)\right) \qquad (t\to +0),$$

(ii) $$a_n > -K\frac{\lambda(n)-\lambda(n-1)}{\lambda(n)}$$

for some $K>0$, *then* $\sum a_n$ *converges to zero.*

4. Last, we illustrate by some special cases derived from our result:

(a) *Let* $0<\alpha<1/2$. *If*

(i) $$\phi_1(t) = o\left(t^{1/(1-\alpha)}\left(\log\frac{1}{t}\right)^{\Delta}\right) \qquad (t\to+0),$$

(ii) $$a_n > -Kn^{\alpha-1}(\log n)^{-\Delta}$$

for some $K>0$, *then* $\sum a_n$ *converges to zero.*

(b) *Let* $0<\alpha<1/2$. *If*

(i) $$\phi_1(t) = o(t^{1/(1-\alpha)}) \qquad (t\to+0),$$

(ii) $$a_n > -Kn^{\alpha-1}$$

then $\sum a_n$ *converges to zero.*

(c) *If*

(i) $$\phi_1(t) = o\left(t\left(\log\frac{1}{t}\right)^{-a}\right) \qquad (t\to+0),$$

(ii) $$a_n > Kn^{-1}(\log n)^{\Delta}$$

for some $\Delta>0$, *then* $\sum a_n$ *converges to zero.*

References

1. G. H. Hardy, *An extension of a theorem on the oscillating series*, Proc. London Math. Soc. vol. 12 (1913) pp. 174–180.

2. G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* (XVII): *Some new convergence criteria for Fourier series*, J. London Math. Soc. vol. 7 (1932) pp. 252–256.

3. ———, *Some new convergence criteria for Fourier series*, Annali della Scuola Normale Superiore di Pisa vol. 3 (1934) pp. 1–20.

4. G. H. Hardy and W. W. Rogosinski, *Fourier series*, Cambridge, 1946, pp. 45, 96.

5. F. C. Hsiang, *On a test for the convergence of Fourier series*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 1036–1039.

6. W. Jurkat, *Zur Convergenztheorie der Fourier-Riehen*, Math. Zeit. vol. 53 (1950) pp. 309–339.

7. K. Kanno, *On the Riesz summability of Fourier series*, Tôhoku Math. J. vol. 8 (1956) pp. 223–234.

8. W. G. Morgan, *On the convergence criteria for Fourier series of Hardy and Littlewood*, Annali della Scuola Normale Superiore di Pisa vol. 4 (1935) pp. 337–382.

9. G. Sunouchi, *Notes on Fourier series*: I, *On the convergence test of Fourier series*, Mathematica Japanica vol. 1 (1948–49) pp. 41–44.

10. F. T. Wang, *On Riesz summability of Fourier series*, Proc. London Math. Soc. (2) vol. 47 (1942) pp. 308–325.

11. ———, *On Riesz summability of Fourier series* (III), Proc. London Math. Soc. (2) vol. 51 (1949) pp. 215–231.

NATIONAL TAIWAN UNIVERSITY

ON THE IDENTITY OF FUNCTION SPACES ON CARTESIAN PRODUCT SPACES[1]

JOHN C. HOLLADAY

For $i=1, \cdots, n$, let S_i be a compact Hausdorff space and F_i be a closed linear subspace of the complex Banach space $C(S_i)$, the set of all continuous functions from S_i to the complex numbers. Let $S_1 \times \cdots \times S_n$ be the Cartesian Product of $S_1, \cdots, S_n$.

Define $F_1 * \cdots * F_n$ as $\{\phi \mid \phi \in C(S_1 \times \cdots \times S_n)$; for any $i=1, \cdots, n$ and $(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$, $\phi(s_1, \cdots, s_{i-1}, \cdot, s_{i+1}, \cdots, s_n) \in F_i\}$. Also, define $F_1 \otimes \cdots \otimes F_n$ as the closure of the space of linear combinations of functions of the form $\phi(s_1, \cdots, s_n) = f_1(s_1) \times \cdots \times f_n(s_n)$ where each $f_i \in F_i$, where we base the topology on the norm, $\|\phi\| = \max_{s_1, \cdots, s_n} |\phi(s_1, \cdots, s_n)|$.

It is easily shown that $F_1 \otimes \cdots \otimes F_n$ is a subspace of $F_1 * \cdots * F_n$ and that if F_2 is one-dimensional, then $F_1 \otimes F_2 = F_1 * F_2$. Furthermore, by using continuous partitions of unity, one may show that $F_1 \otimes C(S_2) = F_1 * C(S_2)$. Therefore, if all but at most one of the F_i are either one-dimensional or $C(S_i)$, then $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$. However, it is not known whether for all cases $F_1 \otimes \cdots \otimes F_n$ will equal $F_1 * \cdots * F_n$ or not.[2] Although this question is not fully answered here, the purpose of this paper is to give a partial answer to this question. The results and arguments of this paper also apply to *real-valued* function spaces.

1. LEMMA. *Let F be a closed linear subspace of $C(S_1)$ and G a closed linear subspace of $C(S_2)$. Let H be a closed subspace of G differing from G by only one dimension. Then $F \otimes G = F * G$ implies that $F \otimes H = F * H$.*

Received by the editors October 22, 1956 and, in revised form, April 16, 1957.

[1] A small part of the work done under an AEC Predoctoral Fellowship at Yale University, year 1952–1953, under the kind and patient guidance of Dr. Charles E. Rickart.

[2] Under Proposition 37 of his first paper of the Amer. Math. Soc. Memoirs, no. 16, Alexandre Grothendieck discusses a number of conjectures which are equivalent to this one.