ON RIESZ SUMMABILITY OF FOURIER SERIES

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1. Let \( \phi(t) \) be an even function, integrable in Lebesgue sense and periodic with period \( 2\pi \). Let

\[
\phi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt.
\]

Let \( \lambda(x) \) be a continuous function, monotone increasing in \( (0, \infty) \) and tends to infinity as \( x \to \infty \). The series \( \sum a_n \) is said to be summable \( (R, \lambda(\omega), r) \) if \( c_r(\omega)/(\lambda(\omega))^r \to s \) as \( \omega \to \infty \), where

\[
c_r(\omega) = \frac{1}{2} (\lambda(\omega) - \lambda(0))^r a_0 + \sum_{n \leq \omega} (\lambda(\omega) - \lambda(n))^r a_n.
\]

Throughout this note, we assume \( s = 0 \), and on writing \( \lambda(x) = e^{\mu(x)} \), \( \mu(x) \) satisfies the following conditions (A):

(i) \( \mu(x) \) is differentiable and monotone increasing for \( x > A \) and \( \mu(x) > 1 \) as \( x \to \infty \),

(ii) \( \mu'(x) \) is monotone decreasing for \( x > A \) and \( \mu'(x) \to 0 \) as \( x \to \infty \),

(iii) \( (\mu'(x))^2 \geq -2\mu''(x) \geq 0 \), \( (\mu'(x))^3 \geq -(3\mu'(x)\mu''(x) + \mu'''(x)) \) for \( x > A \),

(iv) \( U(x) = \int_{x}^{\infty} \frac{du}{u\mu(u)} \) exists for \( x > A \)

and

(a) \( x(\mu'(x))^\Delta U \left( \frac{1}{\mu'(x)} \right) = O(1) \)

for some \( 0 < \Delta < 2 \),

(b) \( x^\Delta U(x) \) being monotone increasing for some \( 0 < \Delta < 2 - \Delta \), \( \mu(x) \) may be taken in the following forms:

1. \( (\log x)^{1+\Delta} \) \( (\Delta > 0, x > A > 1) \),

2. \( x^\alpha \) \( (0 < \alpha < 1/2, x > A > 0) \),

3. \( x^\alpha (\log x)^{-\Delta} \) \( (\Delta > 0, 0 < \alpha < 1/2, x > A > 1) \).

Hardy and Littlewood have derived a convergence test for the oscillating series \( \sum a_n [2; 3; 4] \) which contains two conditions: one is

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concerning the behavior of \( \phi(t) \) near the point \( t = 0 \), e.g., the continuity condition. The other is concerning the magnitude of \( a_n \), e.g., the coefficients condition. Morgan [8] developed Hardy-Littlewood's result so as to permit varying types of conditions on \( \phi(t) \) and \( a_n \). Wang [5; 10; 11] proved Hardy-Littlewood's test in one-sided Tauberian condition for \( a_n \). Sunouchi [9] joined the results of Morgan and Wang and established another type of convergence test for the series \( \sum a_n \). Jurkat [6] developed Sunouchi's result as same as Hardy-Littlewood's to further steps. Very recently, Kanno [7] has extended Wang's theorem to the fractional integral \( \phi_\alpha(t) \) (\( \alpha \geq 1 \)) of \( \phi(t) \). In the present note, we give the above investigations an exhaustive solution. Our result includes, however, to some degrees, the works of the previous authors. It is known that [10], for \( r > 0, \omega > 0, \)

\[
c_r(\omega) = \frac{2}{\pi} \int_0^\infty \phi(t) E_r(\omega, t) dt + o((\lambda(\omega))^r),
\]

where

\[
E_r(\omega, t) = \frac{r}{t} \int_0^\omega (\lambda(\omega) - \lambda(x))^{r-1}\lambda'(x) \sin xtdx.
\]

In addition, we define:

\[
\lambda(x) = \begin{cases} e^{\mu(x)} & (x > A), \\
\lambda(A) + \lambda'(A)(x - A) & (0 \leq x \leq A). \end{cases}
\]

2. The following lemmas are required.

**Lemma 1.** For \( \omega > A \),

\[
E_2(\omega, t) = O(e^{2\mu(\omega)}\omega)
\]

uniformly in \( t \),

\[
E_2(\omega, t) = O(e^{\mu(\omega)}t^{-2}) + O(e^{2\mu(\omega)}\mu'(\omega)t^{-2}),
\]

\[
\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{2\mu(\omega)}\omega t^{-1}),
\]

and

\[
\frac{\partial}{\partial t} E_2(\omega, t) = O(e^{\mu(\omega)}t^{-3}) + O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}).
\]

**Proof.** By the second mean value theorem,\(^1\)

\(^1\) \( B \) is a numerical constant not necessarily the same at each occurrence.
\begin{align*}
E_2(\omega, t) &= \int_0^\omega (\lambda(\omega) - \lambda(x))^2 \cos x \, dx \\
&= (\lambda(\omega) - B)^2 \int_0^{\omega'} \cos x \, dx \\
&= O(e^{2\mu(\omega)}), \\
\frac{\partial}{\partial t} E_2(\omega, t) &= - \int_0^\omega (\lambda(\omega) - \lambda(x))^2 x \sin x \, dx \\
&= - (\lambda(\omega) - B)^2 \omega \int_{\omega'}^\omega \sin x \, dx \\
&= O(e^{2\mu(\omega)}\omega t^{-1})
\end{align*}
for \(\omega > A\), by the second mean value theorem, where \(0 < \omega' < \omega'' < \omega\).

Now,
\begin{align*}
E_2(\omega, t) &= \frac{2}{t} \left( \int_0^A + \int_A^\omega \right) (\lambda(\omega) - \lambda(x))\lambda'(x) \sin x \, dx \\
&= L_1 + L_2.
\end{align*}

\begin{align*}
L_1 &= \frac{2}{t} (\lambda(\omega) - B)\lambda'(A) \int_0^\omega \sin x \, dx = O(e^{\mu(\omega)}t^{-2}), \\
L_2 &= \frac{2}{t} (\lambda(\omega) - B)\lambda'(\omega) \int_{\omega'}^\omega \sin x \, dx,
\end{align*}

since, by (iii) of the conditions (A), \(\lambda'(x)\) is monotone increasing in \((A, \omega)\). It follows that
\begin{align*}
L_2 &= O(e^{2\mu(\omega)}\mu'(\omega)t^{-2}).
\end{align*}

Thus,
\begin{align*}
E_2(\omega, t) &= O(e^{\mu(\omega)}t^{-2}) + O(e^{2\mu(\omega)}\mu'(\omega)t^{-2})
\end{align*}
for \(\omega > A\). Further,
\begin{align*}
\frac{\partial L_1}{\partial t} &= - \frac{2\lambda'(A)}{t^2} \int_0^A (\lambda(\omega) - \lambda(x)) \sin x \, dx \\
&\quad + \frac{2\lambda'(A)}{t} \int_0^A (\lambda(\omega) - \lambda(x)) x \cos x \, dx \\
&= O(e^{\mu(\omega)}t^{-3}).
\end{align*}

By integration by parts once, we find
$$L_2 = \frac{2}{t^2} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At$$

$$+ \frac{2}{t^2} \int_A^\omega \frac{d}{dx} (((\lambda(\omega) - \lambda(x))\lambda'(x)) \cos xtdx$$

$$= \frac{2}{t^2} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At$$

$$- \frac{2}{t^2} \int_A^\omega ((\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x)) \cos xtdx.$$

Denote $H(\omega, x) = (\lambda'(x))^2 - (\lambda(\omega) - \lambda(x))\lambda''(x)$. Then

$$\frac{\partial L_2}{\partial t} = \frac{4}{t^3} (\lambda(\omega) - \lambda(A))\lambda'(A) \cos At + \frac{2A}{t^2} (\lambda(\omega) - \lambda(A)) \sin At$$

$$+ \frac{4}{t^3} \int_A^\omega H(\omega, x) \cos xtdx + \frac{2}{t^2} \int_A^\omega H(\omega, x)x \sin xtdx$$

$$= O(e^{\mu(w)}t^{-3}) + (I_1 + I_2 + I_3 + I_4).$$

$$I_1 = \frac{4}{t^3} \int_A^\omega (\lambda'(x))^2 \cos xtdx = \frac{4}{t^3} (\lambda'(\omega))^2 \int_A^\omega \cos xtdx$$

$$= O((\lambda'(\omega))^2 t^{-3}).$$

$$I_2 = -\frac{4}{t^3} \int_A^\omega (\lambda(\omega) - \lambda(x))\lambda''(x) \cos xtdx$$

$$= -\frac{4}{t^3} (\lambda(\omega) - B)\lambda''(\omega) \int_A^\omega \cos xtdx,$$

since, by (iii) of the conditions (A), $\lambda''(x)$ is monotone increasing in $(A, \omega)$. It follows that

$$I_2 = O(t^{-3}(\lambda(\omega) - B)e^{\mu(\omega)}((\mu'(\omega))^2 + \mu''(\omega))\omega)$$

$$= O(e^{2\mu(\omega)}(\mu'(\omega))^2 t^{-3}) + O(e^{2\mu(\omega)} | \mu''(\omega) | \omega t^{-3}).$$

In view of $(\mu'(x))^2 \geq -2\mu''(x) \geq 0$ for $x > A$, the second term can be absorbed in the first, so that

$$I_2 = O(e^{2\mu(\omega)}(\mu'(\omega))^2 t^{-3}).$$

$$I_2 = \frac{2}{t^2} \int_A^\omega (\lambda'(x))^2 x \sin xtdx$$

$$= \frac{2}{t^2} (\lambda'(\omega))^2 \int_A^\omega \sin xtdx = O((\lambda'(\omega))^2 t^{-3}).$$
\[ I_4 = -\frac{2}{t^2} \int_{\mathcal{A}}^\omega (\lambda(\omega) - \lambda(x))\lambda''(x)x \sin xtdx \]
\[ = -\frac{2}{t^2} (\lambda(\omega) - B)\lambda''(\omega)\omega \int_{\omega'}^\omega \sin xtdx \]
\[ = -\frac{2}{t^2} (\lambda(\omega) - B)e^{\mu(\omega)}((\mu'(\omega))^2 + \mu''(\omega))\omega \int_{\omega'}^\omega \sin xtdx \]
\[ = O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}) + O(e^{2\mu(\omega)}|\mu''(\omega)|\omega t^{-3}) \]
\[ = O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3}). \]

Thus, \((\partial/\partial t)E_2(\omega, t) = O(e^{\mu(\omega)}t^{-3}) + O(e^{2\mu(\omega)}(\mu'(\omega))^2\omega t^{-3})\) for \(\omega > A\). This proves the lemma.

**Lemma 2.** If \(\sum a_n\) is summable \((R, \lambda(\omega), r)\) and
\[ a_n > -K \frac{\lambda(n) - \lambda(n - 1)}{\lambda(n)} , \]
then \(\sum a_n\) converges.

This lemma is known. [1].

3. Now, we are in a position to establish the following

**Theorem 1.** If
\[ \phi_1(t) = \int_0^t \phi(u)du = o\left( t^2 U\left( \frac{1}{t} \right) \right) \quad (t \to +0), \]
then the series \(\sum a_n\) is summable \((R, \lambda(\omega), 2)\) to zero.

**Proof.** We have
\[ \frac{\pi}{2} c_2(\omega) = \int_0^\pi \phi(t)E_2(\omega, t)dt + o(e^{2\mu(\omega)}) \]
\[ = \left( \int_0^{1/A-\epsilon} + \int_{1/A-\epsilon}^\pi \right) \phi(t)E_2(\omega, t)dt + o(e^{2\mu(\omega)}). \]

By the second relation of Lemma 1,
\[ \int_{1/A-\epsilon}^\pi \phi(t)E_2(\omega, t)dt \]
\[ = O(e^{\mu(\omega)}\int_{1/A-\epsilon}^\pi \left| \frac{\phi(t)}{t^2} \right| dt) + O\left( e^{2\mu(\omega)}\mu'(\omega) \int_{1/A-\epsilon}^\pi \left| \frac{\phi(t)}{t^2} \right| dt \right) \]
\[ = O(e^{\mu(\omega)}) + O(e^{2\mu(\omega)}\mu'(\omega)) = o(e^{2\mu(\omega)}), \]
since \( \mu'(\omega) \to 0 \) as \( \omega \to \infty \). By integration by parts,

\[
\int_0^{1/A-\epsilon} \phi(t) E_2(\omega, t) dt = (\int_0^{1/A-\epsilon} \phi_1(t) E_2(\omega, t))_0^{1/A-\epsilon} - \int_0^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt.
\]

The integrated term is easily seen to be \( o(e^{2\mu(\omega)}) \) by the first and the second relations of Lemma 1. Write

\[
\int_0^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt = \left( \int_0^{\mu'(\omega)} + \int_{\mu'(\omega)}^{1/A-\epsilon} \right) \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt
\]

\[= J_1 + J_2.\]

\[
J_1 = o\left(e^{2\mu(\omega)}(\mu'(\omega))^2 \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)
\]

\[= o\left(e^{2\mu(\omega)}(\mu'(\omega))^2 \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)
\]

by (iv) of the conditions (A).

\[
J_2 = \int_{\mu'(\omega)}^{1/A-\epsilon} \phi_1(t) \frac{\partial}{\partial t} E_2(\omega, t) dt
\]

\[= o\left(e^{\mu(\omega)}(\mu'(\omega))-1 \Delta U \left( \frac{1}{\mu'(\omega)} \right) \int_{\mu'(\omega)}^r \left( \frac{1}{\mu'(\omega)} \right) dt \right)
\]

\[+ o\left(e^{2\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \int_{\mu'(\omega)}^r \left( \frac{1}{\mu'(\omega)} \right) dt \right)
\]

\[= o\left(e^{\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right) + o\left(e^{2\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)
\]

\[= o\left(e^{\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right) + o\left(e^{2\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)
\]

\[= o\left(e^{2\mu(\omega)}(\mu'(\omega))^{2-1} \Delta U \left( \frac{1}{\mu'(\omega)} \right) \right)
\]

since \( e^{\mu(\omega)}(\mu'(\omega))^{2} \) is positive and monotone increasing for \( x>A \) by (iii) of the conditions (A). This proves the theorem.

By means of Theorem 1 and Lemma 2, we obtain immediately

**Theorem 2.** If

\[
\phi_1(t) = o\left(t^{\Delta U} \left( \frac{1}{t} \right) \right) \quad (t \to +0),
\]
\[ a_n > - K \frac{\lambda(n) - \lambda(n-1)}{\lambda(n)} \]

for some \( K > 0 \), then \( \sum a_n \) converges to zero.

4. Last, we illustrate by some special cases derived from our result:

(a) Let \( 0 < \alpha < 1/2 \). If

\[ \phi_1(t) = o\left(t^{1/(1-\alpha)} \left(\log \frac{1}{t}\right)^{\Delta}\right) \quad (t \to +0), \]

\[ a_n > - Kn^{\alpha-1}(\log n)^{-\Delta} \]

for some \( K > 0 \), then \( \sum a_n \) converges to zero.

(b) Let \( 0 < \alpha < 1/2 \). If

\[ \phi_1(t) = o(t^{1/(1-\alpha)}) \quad (t \to +0), \]

\[ a_n > - Kn^{\alpha-1} \]

then \( \sum a_n \) converges to zero.

(c) If

\[ \phi_1(t) = o\left(t \left(\log \frac{1}{t}\right)^{-\Delta}\right) \quad (t \to +0), \]

\[ a_n > Kn^{-\Delta}(\log n) \]

for some \( \Delta > 0 \), then \( \sum a_n \) converges to zero.

REFERENCES


ON THE IDENTITY OF FUNCTION SPACES ON CARTESIAN PRODUCT SPACES

JOHN C. HOLLADAY

For $i = 1, \ldots, n$, let $S_i$ be a compact Hausdorff space and $F_i$ be a closed linear subspace of the complex Banach space $C(S_i)$, the set of all continuous functions from $S_i$ to the complex numbers. Let $S_1 \times \cdots \times S_n$ be the Cartesian Product of $S_1, \ldots, S_n$.

Define $F_1 \ast \cdots \ast F_n$ as $\{ \phi \mid \phi \in C(S_1 \times \cdots \times S_n); \text{ for any } i = 1, \ldots, n \text{ and } (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \phi(s_1, \ldots, s_{i-1}, \ldots, s_{i+1}, \ldots, s_n) \in F_i \}$. Also, define $F_1 \otimes \cdots \otimes F_n$ as the closure of the space of linear combinations of functions of the form $\phi(s_1, \ldots, s_n) = f_1(s_1) \times \cdots \times f_n(s_n)$ where each $f_i \in F_i$, where we base the topology on the norm, $\|\phi\| = \max_{s_1, \ldots, s_n} |\phi(s_1, \ldots, s_n)|$.

It is easily shown that $F_1 \otimes \cdots \otimes F_n$ is a subspace of $F_1 \ast \cdots \ast F_n$ and that if $F_2$ is one-dimensional, then $F_1 \otimes F_2 = F_1 \ast F_2$. Furthermore, by using continuous partitions of unity, one may show that $F_1 \otimes C(S_2) = F_1 \ast C(S_2)$. Therefore, if all but at most one of the $F_i$ are either one-dimensional or $C(S_i)$, then $F_1 \otimes \cdots \otimes F_n = F_1 \ast \cdots \ast F_n$. However, it is not known whether for all cases $F_1 \otimes \cdots \otimes F_n$ will equal $F_1 \ast \cdots \ast F_n$ or not. Although this question is not fully answered here, the purpose of this paper is to give a partial answer to this question. The results and arguments of this paper also apply to real-valued function spaces.

1. Lemma. Let $F$ be a closed linear subspace of $C(S_1)$ and $G$ a closed linear subspace of $C(S_2)$. Let $H$ be a closed subspace of $G$ differing from $G$ by only one dimension. Then $F \otimes G = F \ast G$ implies that $F \otimes H = F \ast H$.

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1 A small part of the work done under an AEC Predoctoral Fellowship at Yale University, year 1952–1953, under the kind and patient guidance of Dr. Charles E. Rickart.

2 Under Proposition 37 of his first paper of the Amer. Math. Soc. Memoirs, no. 16, Alexandre Grothendieck discusses a number of conjectures which are equivalent to this one.