A NOTE ON BANACH FUNCTION SPACES

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1. **Introduction.** Let $(X\times Y,\ S\times T,\ \mu\times\nu)$ denote the completion of the Cartesian product of the σ-finite and complete measure spaces (X, S, μ) and (Y, T, ν) [5]. If λ is an arbitrary length function on (X, S, μ), B a Banach space of real valued functions on Y measurable (T), each $f_x=f_x(y)$ in $L^\lambda(B)$ defines a real valued function $f(x, y) =f_x(y)$ on $X\times Y$. The purpose of this note is to study the relation between the spaces $L^\lambda(B)$ where $B=L^\Lambda$, Λ a length function, and the spaces $L^{\lambda\Lambda}$ of functions $f(x, y)$, measurable $(S\times T)$, for which $\lambda\Lambda(f) =\lambda[\Lambda(f_x)]$ defines a norm. Bochner [1] has shown that if X and Y are Euclidean, μ and ν Lebesgue measure and L^λ and L^Λ are the spaces of summable functions on X and Y then $L^{\lambda\Lambda}$ and $L^\lambda(L^\Lambda)$ are equivalent. Our principal result is

THEOREM 1.1. *$L^\lambda(L^\Lambda)$ is always equivalent to a subspace of $L^{\lambda\Lambda}$. In order that $L^\lambda(L^\Lambda)$ be equivalent to $L^{\lambda\Lambda}$ it is necessary and sufficient that either (S) be σ-atomic* (λ) *or* (L9), (L12) *and* (L13) *all hold for* Λ.

2. **Terminology.** The notation $f(x, y)$ will refer to a real valued function defined on $X\times Y$, $f_x=f_x(y)$ to a function on X valued almost everywhere in a space of functions measurable (T) or to an X-section of $f(x, y)$. If Q, Q_a are subsets of $X\times Y$, Q_x, $Q_{a,x}$ refer to the sections of Q, Q_a determined by x [5, p. 141]. By $f_n(P)\uparrow f(P)$ is meant $f_1(P) \leqq f_2(P)\leqq \cdots$ and $f(P)=\sup_n f_n(P)\leqq\infty$ for all points P. Measurability symbols (S), (T), $(S\times T)$ will be omitted when no confusion is likely to result. The definition of simple function will differ from [5, p. 84] in that the constants will sometimes be vectors. We shall write χ_E, $\lambda(E)$ as abbreviations for the characteristic function of E and $\lambda(\chi_E)$, respectively, f_E for $f\chi_E$.

As in [3] λ will be called a length function on (X, S, μ) if for every measurable function u with $0\leqq u(x)\leqq\infty$ for almost all x, $\lambda(u)$ is defined with $0\leqq\lambda(u)\leqq\infty$ and satisfies: (L1) $\lambda(u)=0$ if $u(x)=0$ almost everywhere, (L2) $\lambda(u)\leqq\lambda(u_1)$ whenever $u(x)\leqq u_1(x)$ for all x, (L3) $\lambda(u_1+u_2)\leqq\lambda(u_1)+\lambda(u_2)$, (L4) $\lambda(ku)=k\lambda(u)$ for all $k>0$, and (L5) $\lambda(u)=\sup\lambda(u_n)$ if $u_n\uparrow u$. In addition we shall sometimes impose some or all of conditions: (L9) Either X is coarse or for every E with $\lambda(E)<\infty$, $\lambda(e)\to 0$, whenever $\mu(e)\to 0$, $e\subset E$ [3, p. 592], (L12)

Received by the editors December 13, 1956 and, in revised form, April 30, 1957.

[1] This work was partially supported by the Office of Ordnance Research, U. S. Army, under Contract No. DA-36-034-ORD-1622, Supplemental Agreement No. 2.

$\lambda(u)<\infty$, $\epsilon>0$ imply $\lambda(u-u_e)<\epsilon$ for some e with $\lambda(e)<\infty$, [**3**, p. 592] and (L13) $\lambda(u)<\infty$ implies that $\lambda(u-u_N)\rightarrow 0$ as $N\rightarrow\infty$ where $u_N(x)=\min(N, u(x))$ [**3**, p. 577]. We shall suppose throughout that L^λ and L^Λ contain at least one element different from zero. We also suppose that X and Y contain no λ, Λ-null sets of positive measure. The theorems remain true when such sets exist, the arguments given below holding after they have been deleted.

If $\mu(E)<\infty$, E will be called σ-atomic if the union of the atomic subsets of E (at most countable in number) differs from E by a null set. If every $E\in S$ with $\mu(E)<\infty$, $0<\lambda(E)<\infty$ is σ-atomic we say that (S) is σ-atomic (λ).

The space of functions f_x, valued in the Banach space B, Bochner measurable outside a maximal λ-null set and with $\lambda(f_x)=\lambda(|f_x|)<\infty$, will be denoted by $\mathfrak{L}^\lambda(B)$, the associated Banach space (with points equivalence classes in $\mathfrak{L}^\lambda(B)$) by $L^\lambda(B)$. *When* $\lambda\Lambda(f)=\lambda[\Lambda(f_x)]$ *is defined for every* $f(x, y)$, *measurable* $(S\times T)$, $\mathfrak{L}^{\lambda\Lambda}$ will denote the space of measurable $f(x, y)$ with $\lambda\Lambda(f)<\infty$, $L^{\lambda\Lambda}$ the associated space.

3. Spaces of functions measurable $(S\times T)$.

THEOREM 3.1. *$L^{\lambda\Lambda}$ is a Banach space.*

PROOF. Because of [**3**, Theorem 3.1], we need only show that $\lambda\Lambda$ is a length function when λ and Λ are length functions (assuming that $\lambda\Lambda(f)$ is defined for every measurable f). That (L1)–(L4) hold is easily shown. We verify that (L5) holds for $\lambda\Lambda$. Suppose that $f_n(x, y)\uparrow f(x, y)$ where each f_n is measurable $(S\times T)$. By [**5**, 34.B, 36.4] $f_{n,x}(y)$ is measurable (T) for all n and almost all x. If x is such that $f_{n,x}$ is measurable (T) for all n, $\Lambda(f_x)=\sup\Lambda(f_{n,x})$ by (L5) for Λ. There is thus a set N which is a countable collection of null sets, with $\Lambda(f_{n,x})\uparrow\Lambda(f_x)$ in $X-N$. (L5) and the fact that $\lambda(g-h)=0$ if g and h differ in a null set then implies that

$$\lambda\Lambda(f)=\lambda\left[\sup_n\Lambda(f_{n,x})\right]=\sup_n\lambda\Lambda(f_{n,x}).$$

If $\lambda\Lambda$ is to be a length function it must, by definition, be defined for every $f(x, y)$ that is measurable $(S\times T)$. On the other hand if $f(x, y)$ is measurable $(S\times T)$, $f_x=f_x(y)$ is measurable (T) for almost all x so that $\Lambda(f_x)$ is a function of x defined for almost all x. If $\Lambda(f_x)$ is measurable (S), $\lambda\Lambda(f)=\lambda[\Lambda(f_x)]$ will then be defined for every f measurable $(S\times T)$.

THEOREM 3.2. *If $f(x, y)$ is measurable $(S\times T)$, $\Lambda(f_x)$ is measurable (S) if*: (i) *for each $a\geqq 0$, $E_a=\{x:\Lambda(f_x)>a\}$ differs by a null set from*

$E_a^0 = \{x: \nu(Q_{a,x}) > 0\}$, *where* $Q_a = \{(x, y): |f(x, y)| > a\}$; *or* (ii) (L9) *holds for* Λ.

PROOF. (i) Since Q_a is measurable $(S \times T)$, $\nu(Q_{a,x})$ is measurable (S) [**5**, 35.A, 36.4] so that E_a^0 and, by hypothesis, E_a are measurable (S) for every a and therefore $\Lambda(f_x)$ is measurable (S). An example of a length function for which (i) holds for every measurable f is given by

$$\Lambda[g(y)] = \operatorname*{ess. sup.}_{y \in Y} g(y).$$

In the proof of (ii) we shall use

(A) If $E \subset X \times Y$ is measurable $(S \times T)$ then $E \subset \bigcup_1^\infty A_i \times B_i$ where $\mu(A_i) < \infty$, $\nu(B_i) < \infty$, $i = 1, 2, \cdots$ and the measurable "rectangles" $A_i \times B_i$ are disjoint and not empty.

(B) If $\mu \times \nu(E) < \infty$, $\epsilon > 0$, there exists $H = \bigcup_1^n A_i \times B_i$, $\mu(A_i) < \infty$, $\nu(B_i) < \infty$, $i = 1, 2, \cdots, n$, with $\mu \times \nu(E \Delta H) < \epsilon$, where Δ denotes symmetrical difference.

(C) If $f_n(x, y) \uparrow f(x, y)$ and $\Lambda(f_{n,x})$ is measurable for all n, then $\Lambda(f_x)$ is measurable.

The definition of $(X \times Y, S \times T, \mu \times \nu)$ implies (A) and (B) and (L5) for Λ and standard measurability theorems give (C).

(ii) We first suppose that $\mu(X) < \infty$, $\nu(Y) < \infty$, $\Lambda(Y) < \infty$. If then $f(x, y) = \chi_Q$, where Q is measurable $(S \times T)$, there is by (B) a sequence $\{H_n\}$ of finite sums of finite measurable "rectangles" $\bigcup_i (A_{ni} \times B_{ni}) = H_n$ with $\mu \times \nu(Q \Delta H_n)$ approaching zero as $n \to \infty$. Each $\Lambda(H_n)_x$ is simple and therefore measurable (S). For a sequence $\epsilon_n > 0$, $\epsilon_n \to 0$, let H_n be chosen so that $\mu \times \nu\ (Q \Delta H_n) < \epsilon_n^2$. If $E_n = \{x: \nu(Q \Delta H_n)_x > \epsilon_n\}$, $\mu(E_n) < \epsilon_n$ since $\mu \times \nu(Q \Delta H_n) = \int_X (Q \Delta H_n)_x d\mu(x)$ [**5**, 35.B]. Thus $|\Lambda(Q_x) - \Lambda(H_{n,x})| \leqq \Lambda(Q \Delta H_n)_x$ where, by (L9), $\sup \Lambda(Q \Delta H_n)_x \to 0$, $x \in X - E_n$, and $\mu(E_n) \to 0$ as $n \to \infty$. There is thus a subsequence of $\{\Lambda(H_n)_x\}$ converging almost uniformly to $\Lambda(Q_x)$, which is therefore measurable. If $f(x, y)$ is simple the argument is similar, each set Q_i on which f is constant being approximated by sequences of collections of rectangles. The extension to bounded and to arbitrary measurable functions is routine using (C). The proof is then complete when $\mu(X) < \infty$, $\nu(Y) < \infty$, $\Lambda(Y) < \infty$.

Suppose that $\nu(Y) < \infty$, $\Lambda(Y) = \infty$ and let K denote the family of sets $e \in T$, modulo null sets, with $\nu(e) > 0$, $\Lambda(e) < \infty$. If e_α, e_β are in K so is their union. (K will be empty if there is no function $g(y)$ measurable (T) with $0 < \Lambda(g_Y) < \infty$.) K is partially ordered by inclusion. If $\mathfrak{N}$ is a nest in K, by the Hausdorff maximal principle [**7**, p. 32], there is a maximal nest $\mathfrak{M}$ in K which contains $\mathfrak{N}$. If $e \subset Y - \bigcup\{M: M \in \mathfrak{M}\}$ and $\nu(e) > 0$, then $\Lambda(e) = \infty$, since otherwise $\{M \cup e\}$ would be a nest properly containing $\mathfrak{M}$. For $i=1, 2, \cdots$ let Y_i be an arbitrary set in $\mathfrak{M}$ with $\nu(Y_i)>(1-1/2^i)\ \sup_{M\in\mathfrak{M}}\nu(M)$. Then $\nu(\bigcup Y_i)=\sup \nu(M)\leqq\nu(Y)$ and

$$(3.1)\quad Y = Y_0 + \bigcup_1^\infty Y_i,\quad \Lambda(e) = \infty \text{ if } e\subset Y_0,\quad \nu(e) > 0;\quad \Lambda(Y_i) < \infty,$$

$i=1, 2, \cdots$ (cf. [8, §§2, 3]). If $f(x, y)$ is measurable $(S\times T)$, (A) and the σ-finiteness of ν permit us to assume that $f(x, y)=0$ for y outside $Y=\bigcup_1^\infty B_i$ with $\nu(B_i)<\infty$ and combining the decompositions (3.1) for all B_i shows that (3.1) holds for Y. Since μ is also σ-finite we can assume that $f(x, y)$ vanishes outside $X=\bigcup_1^\infty X_i$ with $\mu(X_i)<\infty$ and that, for $H_n=\bigcup_1^n X_i\times Y_i$, $Y'=Y-Y_0$, $f_{H_n}\uparrow f_{X\times Y'}$ as $n\to\infty$ so that $f_{X\times Y'}$ is measurable by the first part of the proof and (C). Finally if $E_x=\{y\in Y_0: f(x, y)>0\}$, $\Lambda(f_x)=\infty$ if $\nu(E_x)>0, =\Lambda(f_{X\times Y'})$ if $\nu(E_x)=0$. Since $\{x: \nu(E_x)>0\}$ is measurable, $\Lambda(f_x)$ is measurable.

REMARK. If λ and Λ determine L^1 norms, Theorem 3.2 is part of the Fubini theorem.

If $f(x, y)$ is measurable $(S\times T)$, f_x is valued almost everywhere in the space of real functions measurable (T). If $f\in\mathfrak{L}^{\lambda\Lambda}$, f_x is valued almost everywhere in $\mathfrak{L}^\Lambda$, $\lambda(f_x)=\lambda\Lambda(f)<\infty$ and $f_x\in\mathfrak{L}^\lambda(L^\Lambda)$ if and only if f_x is Bochner measurable.

LEMMA 3.1. *In order that, for every* $f(x, y)\in\mathfrak{L}^{\lambda\Lambda}$, $f_x\in\mathfrak{L}^\lambda(L^\Lambda)$ *it is necessary and sufficient that either* (i) (S) *is* σ*-atomic* (λ) *or* (ii) *satisfies* (L9), (L12) *and* (L13).

PROOF. Assume that (i) holds. The definition of $(X\times Y, S\times T, \mu\times\nu)$ implies that, if $f(x, y)$ is measurable $(S\times T)$ and $S_0\in S$ is atomic, there is a function $\bar{f}(x, y)$ with $\bar{f}_x$ constant in S_0 and with $\mu\times\nu(S_1)=0$ where $S_1=\{(x, y): f(x, y)\neq\bar{f}(x, y)\}$ (cf. [5, 16.4, p. 72]).

Let $f\in\mathfrak{L}^{\lambda\Lambda}$ and suppose that $\mu(x)<\infty$. If $X_n=\{x: \Lambda(f_x)>1/n\}$, then $X_0=\{x: \Lambda(f_x)>0\}=\bigcup_1^\infty X_n$, $\mu(N)=0$ where $N=\{x: \Lambda(f_x)=\infty\}$, and $\mu(X_0-X_n)\to 0$ as $n\to\infty$. For each n,

$$0 \leqq \Lambda(X_n) \leqq \Lambda(nf_x) \leqq n\Lambda(f_x) < \infty, \qquad x \in X_0 - N.$$

Since X is σ-atomic (λ) each $X_n=\bigcup_{i=1}^\infty A_{ni}+N_n$ where each A_{ni} is atomic and $\mu(N_n)=0$. Then f_x is equivalent $(S\times T)$ to $\bar{f}_x$ which is constant on each A_{ni}. The proof that $\bar{f}_x$ and f_x are Bochner measurable on X is then trivial. The extension to an arbitrary X is then a consequence of the σ-finiteness of μ.

We next show that (ii) is sufficient. Since μ is σ-finite we can assume that $\mu(X)<\infty$. In the first part of the proof of Theorem 3.2 (ii) each $\chi_{H_n,x}$ is simple and it follows easily that χ_{Q_x} is Bochner measurable.

The extension to f_x corresponding to a bounded, measurable $f(x, y)$ is easy. If $f(x, y)\in \mathfrak{L}^{\lambda\Lambda}$ is not bounded, $\lambda\Lambda(f)<\infty$ implies that $\Lambda(f_x)<\infty$ outside a μ-null set $X_0\subset X$. (L13) then implies that $\Lambda(f-f_N)_x\to 0$ as $N\to\infty$, $x\notin X_0$. For $\epsilon>0$ let $E_N=\{x: \Lambda(f-f_N)_x>\epsilon\}$. For N sufficiently large $\mu(E_N)<\epsilon$ since $\mu(X)<\infty$. This, with the Bochner measurability of all the f_N implies that f is Bochner measurable when $\nu(Y)<\infty$, $\Lambda(Y)<\infty$. In the general case we can assume from (A) that $f(x, y)=0$ outside $\bigcup_1^\infty B_i$ where $\nu(B_i)<\infty$, $\Lambda(B_i)<\infty$, $i=1, 2, \cdots$. Then $f_x^m=f_x\chi_{\bigcup_1^m B_i}$ is Bochner measurable for every m. For each x outside a λ-null set, where e and N are as in (L12) and (L13),

$$\begin{aligned}\Lambda(f_x-\overset{m}{f_x}) &\leqq \Lambda(f_x-f_{N,x})+\Lambda(\overset{m}{f}_{N,x}-\overset{m}{f_x})+\Lambda(f_{N,x}-f_{N,x}\chi_e)\\ &\quad+\Lambda(f_{N,x}-\overset{m}{f}_{N,x})_e+\Lambda(\overset{m}{f}_{N,x}-\overset{m}{f}_{N,x}\chi_e)\\ &\leqq 2\Lambda(f_x-f_{N,x})+2\Lambda(f_{N,x}-f_{N,x}\chi_e)+N\Lambda\left(e\bigcup_m^\infty B_i\right),\end{aligned}$$

which can be made arbitrarily small by choice of N, e and m, using (L12) and (L13). That f_x is Bochner measurable then follows easily as in the preceding paragraph.

Finally we show that at least one of (i), (ii) is necessary to ensure that, for every $g(x, y)\in\mathfrak{L}^{\lambda\Lambda}$, g_x is Bochner measurable. Suppose that S contains a set S_1 with $\mu(S_1)<\infty$, $0<\lambda(S_1)<\infty$ and suppose that S_1 contains no atoms. Then S contains a sequence S_n of sets satisfying $S_1=S_2\cup S_3$, $S_2=S_4\cup S_5$ and in general $S_i=S_j\cup S_{j+1}$ for some $j>i$, where $\mu(S_i)>0$, $i=1, 2, \cdots$ and $\mu(S_i)\to 0$ as $i\to\infty$ (cf. [**4**, §2]). Suppose that there is a non-negative function $f(y)\in L^\Lambda$, a number $\delta>0$, and a collection of disjoint sets $\{T_i\}$ in T with $\Lambda(f_{T_i})>\delta$, $i=1, 2, \cdots$. Define $g(x, y)=f_{T_i}(y)=f\chi_{T_i}$ for all $x\in S_i$, $i=1, 2, \cdots$, $g(x, y)=0$ elsewhere. Then clearly $g(x, y)$ is measurable $(S\times T)$ and, since $\Lambda(g_x)\leqq\Lambda(f_x)$ for all $x\in S_1$, $\Lambda(g_x)=0$ elsewhere, $\lambda\Lambda(g)\leqq\Lambda(f)\lambda(S_1)<\infty$ so that $g\in\mathfrak{L}^{\lambda\Lambda}$. If for some n, S_n contains x but not $x'\in S_1$,

$$|g_x(y)-g_{x'}(y)|\geqq f_{T_i}(y),\qquad \Lambda(g_x-g_{x'})\geqq\Lambda(f_{T_i})>\delta.$$

Thus if x is fixed and $\{S_{n_i}\}$ is the subsequence of $\{S_n\}$ containing x, the set of points x' for which $\Lambda(g_x-g_{x'})<\delta$ is contained in $\bigcap S_{n_i}$ and therefore has measure zero. If $\bar{g}_x$ is an arbitrary simple function on X, $\Lambda(g_x-\bar{g}_x)>\delta/2$ for almost all $x\in S_1$ so that g_x cannot be Bochner measurable. The argument of [**4**, Lemma 3.2] shows that there always exists such a function $f(y)\in L^\Lambda$, $\delta>0$ and $\{T_i\}$ if one of (L9), (L12) and (L13) fails to hold so that the above construction gives a counterexample if (i) and one of (L9), (L12) and (L13) fail to hold.

4. **The equivalence of the spaces $L^{\lambda\Lambda}$ and $L^{\lambda}(L^{\Lambda})$.** Where μ, ν denote Lebesgue measure on $(0, 1)=X=Y$, Sierpinski [9] (using the axiom of choice) has constructed a plane set Q having at most two points in common with every parallel to the axes and not measurable $(S\times T)$. For the function χ_Q, $\Lambda(Q_x)=0$ for all x so that χ_{Q_x} is Bochner measurable and in $\mathfrak{L}^{\lambda}(L^{\Lambda})$ but not in $\mathfrak{L}^{\lambda\Lambda}$ since it is not measurable. This example shows that the statement of Theorem I in [1] is not precise. We note that when the conditions of Lemma 3.1 are not satisfied (for example if $L^{\Lambda}=L^{\infty}$, and x, y, μ, ν are as in this paragraph) $\mathfrak{L}^{\lambda\Lambda}$ and $\mathfrak{L}^{\lambda}(L^{\Lambda})$ can each contain elements with the corresponding element not in the other. We note that for Sierpinski's example there is a function (namely $g_x\equiv 0$) equivalent to f_x in $L^{\lambda}(L^{\Lambda})$ with $g(x, y)$ measurable $(S\times T)$. That this is always the case is shown by the following lemma which generalizes part of Bochner's Theorem I [1].

LEMMA 4.1. *If $f_x\in L^{\lambda}(L^{\Lambda})$ then $\hat{f}_x$, the equivalence class of f_x, contains g_x with $g(x, y)\in L^{\lambda\Lambda}$.*

PROOF. First suppose that $\mu(X)<\infty$, $\lambda(X)<\infty$. For $\epsilon>0$ the Bochner measurability of f_x implies the existence of a set $E_1\subset X=E_0$ with $\mu(E_1)<\epsilon$ and a sequence of simple functions f_x^i valued in L^{Λ} with $\Lambda(f_x-f_x^i)\rightarrow 0$ uniformly in E_0-E_1. Each $f^i(x, y)\in L^{\lambda\Lambda}$ and the f^i, restricted to E_0-E_1, form a Cauchy sequence in $L^{\lambda\Lambda}$ and define a function $g_1(x, y)\in L^{\lambda\Lambda}$, vanishing in $E_1\times Y$, with $\lambda(f_x-g_{1,x})_{E_0-E_1}=0$. Replacing E_i by E_{i+1}, $i=1, 2, \cdots$, there is a sequence of sets E_i with $E_i\subset E_{i+1}$, $\mu\bigcup_0^{\infty}(E_i-E_{i+1})=\mu(X)$, and a corresponding sequence of functions $g_i(x, y)\in L^{\lambda\Lambda}$, vanishing outside $E_{i-1}-E_i$ with

$$\lambda(f_x - g_{i,x})_{E_{i-1}E_i} = 0. \tag{4.1}$$

Set $g(x, y)=\sum_1^{\infty} g_i(x, y)$. Then $g(x, y)\in L^{\lambda\Lambda}$, $g_x\in\mathfrak{L}^{\lambda}(L^{\Lambda})$ and, since $(f_x-g_x)_{\bigcup_1^n E_i-E_{i+i}}\uparrow(f_x-g_x)$ outside a null set, (4.1) and (L5) for λ show that $g_x\in\hat{f}_x$.

In the general case the set $X_0=\{x:\Lambda(f_x)>0\}$ is measurable (S). If $X_n=\{x:\Lambda(f_x)>1/n\}$, $X_0=\bigcup_1^{\infty}X_n$ and $\lambda(X_n)<\infty$, $n=1, 2, \cdots$. Since X_0 is σ-finite we can assume that $\mu(X_n)<\infty$ and that the X_n are disjoint. An argument similar to that given in the first part then completes the proof.

Lemmas 3.1 and 4.1 combine to give Theorem 1.1. Theorem 1.1 has the following corollaries.

COROLLARY 1. *If L^{Λ} is separable or reflexive, $L^{\lambda\Lambda}$ is equivalent to $L^{\lambda}(L^{\Lambda})$.*

COROLLARY 2. *$L^{\lambda\Lambda}$ is separable (reflexive) if and only if L^{λ} and L^{Λ} are separable (reflexive).*

If L^{Λ} is separable (L9), (L12) and (L13) hold by [**4**, Lemma 3.2]. If L^{Λ} is reflexive they hold by [**6**, p. 206]. If $L^{\lambda\Lambda}$ is separable (reflexive), L^{Λ} is separable (reflexive) since it is isometric to a subspace of $L^{\lambda\Lambda}$. Then Corollary 1, [**4**, Lemma 3.2] and [**6**, Theorem 1.1] give Corollary 2.

The basis problem for the spaces $L^{\lambda\Lambda}$ is equivalent to the basis problem for the spaces $L^{\lambda}(L^{\Lambda})$ and is therefore partially solved by [**2**]. That the results of [**4**] do not apply directly to $L^{\lambda\Lambda}$ is shown by the fact that even when both λ and Λ are levelling (i.e. have property (L8) of length functions [**3**]) $\lambda\Lambda$ need not be leveling or even have the weaker property (3.4) of [**4**]. This is shown by the following elementary example. Let $X = Y = (0, \infty)$, let S, T be the Lebesgue measurable sets, μ and ν Lebesgue measure on $(0, \infty)$ and let $L^{\lambda} = L^2$, $L^{\Lambda} = L^1$. Let $R_n = \{(x, y): n-1 < x \leqq n;\ n-1 < y \leqq n\}$, $R'_n = \{(x, y): 0 < x < 1,\ n-1 < y \leqq n\}$. Let $f_k(x, y)$ denote the characteristic function of $\bigcup_1^k R_n$, Af_k the average of f_k over $R_n \cup R'_n$ in $R_n \cup R'_n$, $n = 1, 2, \cdots$; $= f_k$ elsewhere. Then every $f_k \in L^{\lambda\Lambda}$ but actual computation shows that $\lambda\Lambda(Af_k)/\lambda\Lambda(f_k) > k^{1/2}/2$.

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