ON A THEOREM BY A. E. TAYLOR

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Let $B$ be a complex normed linear space. It is well known [1] that, for any bounded linear functional $\phi$ defined on a linear subspace $M$ of $B$, there exists a norm-preserving linear extension $f$ of $\phi$ to $B$, i.e. a bounded linear functional $f$ defined on $B$ such that (i) $f(x) = \phi(x)$ for all $x \in M$, (ii) $\|f\|_B = \|\phi\|_M$, where $\|f\|_B$ and $\|\phi\|_M$ denote the norms of bounded linear functionals $f$ and $\phi$ on $B$ and $M$, respectively. It was proved by A. E. Taylor [2] that if the conjugate space of $B$ is strictly convex, then $f$ is uniquely determined by $\phi$. The purpose of this note is to show that the converse of this theorem is true, i.e. we want to prove the following

**Theorem.** Let $B$ be a complex normed linear space whose conjugate space is not strictly convex. Then there exists a bounded linear functional defined on a linear subspace of $B$ for which a norm preserving linear extension to $B$ is not unique.

**Proof.** Let $f_1$ and $f_2$ be two bounded linear functionals on $B$ such that (i) $f_1 \neq f_2$, (ii) $\|f_1\|_B = \|f_2\|_B = \|(f_1 + f_2)/2\|_B = 1$. Let us put $M = \{x \mid f_1(x) = f_2(x)\}$ and $\phi(x) = f_1(x) = f_2(x)$ on $M$. It suffices to prove that $\|\phi\|_M = 1$. Let $z$ be an element of $B$ such that $f_1(z) - f_2(z) = 1$. Then every element $x$ of $B$ can be uniquely expressed in the form: $x = y + az$, where $y \in M$ and $a = f_1(x) - f_2(x)$ is a complex number. Let $\{x_n \mid n = 1, 2, \cdots\}$ be a sequence of elements of $B$ such that $\|x_n\| = 1$ for $n = 1, 2, \cdots$ and $\lim_{n \to \infty} (f_1(x_n) + f_2(x_n))/2 = 1$. Then it is easy to see that $\lim_{n \to \infty} f_1(x_n) = \lim_{n \to \infty} f_2(x_n) = 1$. Thus, if we put $x_n = y_n + a_n z$, where $y_n \in M$ and $a_n = f_1(x_n) - f_2(x_n)$, $n = 1, 2, \cdots$, then $\lim_{n \to \infty} a_n = 0$, and hence

$$\lim_{n \to \infty} \|y_n\| = \lim_{n \to \infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \phi(y_n) = \lim_{n \to \infty} f_1(y_n) = 1.$$  

From this follows that $\|\phi\|_M \geq 1$ and hence $\|\phi\|_M = 1$.

**References**


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