1. Introduction. A combinatorial theorem \([1; 3]\) usually referred to as “the marriage problem” or “the problem of distinct representatives” has the following matrix formulation; the convex hull of the set of all \(n\) by \(n\) permutation matrices is the set of all \(n\) by \(n\) doubly stochastic matrices. In this note the above theorem is generalized.

The following notation and definitions will be used. \(A\) will represent an \(n\) by \(n\) matrix with non-negative real entries \(a_{ij}\); \(S\) will represent the sum of all entries of \(A\), \(S = \sum_i \sum_j a_{ij}\); \(R_i\) will represent the sum of the entries in the \(i\)th row and \(C_j\) will represent the sum of the entries in the \(j\)th column; \(M\) will represent the largest row or column sum of \(A\), \(M = \max (R_i, C_j)\). Also used will be the concept of a sub-permutation matrix of rank \(r\). By this is meant a matrix \(P\) with the following properties: (1) each entry of \(P\) is either 1 or 0; (2) each row and each column of \(P\) contains at most one 1; (3) \(P\) contains exactly \(r\) entries equal to 1. In terms of this notation the theorem quoted above becomes; a matrix \(A\) lies in the convex hull of the set of all permutation matrices if and only if \(M = 1\) and \(S = n\). In \([2]\) the authors of the present note obtain sufficient conditions in order that a matrix \(A\) with non-negative entries contain nonzero entries in the places occupied by 1 in a permutation matrix of rank \(r\). In this note necessary and sufficient conditions are given in order that a matrix \(A\) lie in the convex hull of the sub-permutation matrices of rank \(n - i\) (\(i = 0, 1, 2, \ldots, n - 1\)).

2. The Theorem. Let \(A\) be an \(n\) by \(n\) matrix whose entries are non-negative real numbers. A necessary and sufficient condition that \(A\) lie in the convex hull of all sub-permutation matrices of rank \(n - i\) is that \(S = n - i\) and \((n - i)/n \leq M \leq 1\).

Proof. The necessity is obtained as follows. Let \(A = \sum_j \alpha_j P_j\) where \(\alpha_j \geq 0\), \(\sum_j \alpha_j = 1\) and \(P_j\) is a sub-permutation matrix of rank \(n - i\). Then each matrix \(\alpha_j P_j\) has the sum of all its entries equal to \((n - i)\alpha_j\) and each row or column sum has the value \(\alpha_j\) or 0. Hence \(S = (n - i) \sum_j \alpha_j = (n - i)\) and \(M = \sum_j \alpha_j = 1\). Also since \(n - i = S = \sum_j R_j \leq nM\), \((n - i)/n \leq M\). Hence \(S = n - i\) and \((n - i)/n \leq M \leq 1\).

To obtain the sufficiency we note that if \(S = n - i\) and \((n - i)/n \leq M \leq 1\)
\[ \sum R_j = \sum C_j = n - i. \] Also the numbers \( 1 - R_1, 1 - R_2, \ldots, 1 - R_n \) are non-negative and at least one of these is positive if \( i > 0 \). For if all of \( 1 - R_1, 1 - R_2, \ldots, 1 - R_n \) were 0 then \( R_j = 1 = M \) for all \( j \) so that \( S = n \) a contradiction. The matrix \( A \) is now augmented to a matrix \( A^* \) by the addition of \( i \) rows and \( i \) columns as follows: 

- \( a_{rs}^* = a_{rs} \) if \( r \) and \( s \) are both less than or equal to \( n \);
- \( a_{rs}^* = 0 \) if \( r \) and \( s \) are both greater than \( n \);
- \( a_{n+r, s}^* = (1 - R_s)/i \) for \( r = 1, 2, \ldots, n; t = 1, 2, \ldots, i; a_{n+u,v}^* = (1 - C_v)/i \) for \( u = 1, 2, \ldots, i; v = 1, 2, \ldots, n \).

The matrix \( A^* \) is a doubly stochastic \( n+i \) by \( n+i \) matrix with zeros in the lower right hand \( i \) by \( i \) block. By the theorem quoted in the introduction \( A^* = \sum \alpha_r P_r^* \) where \( \alpha_r \geq 0, \sum \alpha_r = 1 \) and \( P_r^* \) is an \( n+i \) by \( n+i \) permutation matrix. Furthermore, each \( P_r^* \) has an \( i \) by \( i \) block of zeros in its lower right corner. Hence \( P_r^* \) has 2\( i \) entries equal to 1 in its last \( i \) rows and \( i \) columns. If \( P_r \) is the \( n \) by \( n \) matrix in the upper left hand corner of \( P_r^* \), \( P_r \) contains \( (n+i) - 2i = n - i \) ones. Hence \( P_r \) is a sub-permutation matrix of rank \( n - i \). Also \( A = \sum \alpha_r P_r \).

**Bibliography**


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