ON HARMONIC MAPPINGS

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1. Suppose that the functions \( x = x(\alpha, \beta) \), \( y = y(\alpha, \beta) \) define a one-to-one harmonic mapping of the unit disc \( \Gamma \) in the \( \alpha, \beta \)-plane \( (\alpha + i\beta = \gamma) \) onto a convex domain \( C \) in the \( x, y \)-plane \( (x + iy = z) \). The origin is assumed to be fixed. Introducing two functions \( F(\gamma) \) and \( G(\gamma) \) which, in \( \Gamma \), depend analytically upon the variable \( \gamma \) we may write \( z = \text{Re} \ F(\gamma) + i \text{Re} \ G(\gamma) \). The purpose of the present paper is (i) to give a new proof of a lemma which, in a special form, was first used by T. Rado \[13\] and which was proved in general by L. Bers (see \[2, \text{Lemma 3.3}\]),\(^2\) (ii) to derive an improved value for an important constant first introduced by E. Heinz \[3\]. The proofs will be very simple due to the fact that there is a close connection between univalent harmonic mappings and the minimal surface equation (see e.g. \[11, \text{footnote 2}\]) and also the differential equation

\[
\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1.
\]

The connection with the latter equation was exploited by K. Joergens \[8\] for the study of the solutions of \( (1) \). One can, however, proceed one step further by introducing a mapping which was invented by H. Lewy \[10\] for Monge-Ampère equations.

2. Let \( z = \text{Re} \ F(\gamma) + i \text{Re} \ G(\gamma) \) be a harmonic mapping with the properties mentioned above. Then the expression

\[
\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1.
\]

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\(^2\) It has been shown by H. Hopf (cf. \[7, \text{p. 133 and 5, pp. 91–92}\]) that the combination of Heinz's inequality with Schwarz's lemma yields a sharper result.
\[ \phi = \frac{1}{2} \text{Im} \left( F \overline{G} + \int_0^\gamma (FG' - F'G) d\gamma \right) \]

may be regarded as a function \( \phi(x, y) \) of \( x \) and \( y \), defined in \( C \) (see K. Joergens \[8, p. 339\]). By a straightforward computation it can be verified that \( \phi(x, y) \) is a solution of the Equation (1). In fact, one obtains

\[
\begin{align*}
\rho &= \text{Im} \ G(y), & q &= \text{Im} \ F(y), \\
\tau &= |G'|^2 \cdot |\text{Im} \ F'G'|^{-1}, & s &= -|\text{Re} \ F'G'| \cdot |\text{Im} \ F'G'|^{-1}, \\
t &= |F'|^2 \cdot |\text{Im} \ F'G'|^{-1}.
\end{align*}
\]

Here \( \rho, q, r, s, t \) are abbreviations for \( \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{uv} \), as usual. According to a lemma of H. Lewy \[9\], \( \text{Im} (F'G') = x_0y_0 - x_0y_0 \neq 0 \) in \( \Gamma \). It may be assumed that \( \text{Im} (F'G') > 0 \). Then \( \phi_{xx} > 0 \). Now consider, in \( C \), the functions

\[ u = u(x, y) = x + \rho(x, y), \quad v = v(x, y) = y + q(x, y) \]

and put \( u + iv = w \). For any two points \( z_1 \) and \( z_2 \) in \( C \) the following inequality holds true

\[ (x_2 - x_1)[\rho(x_2, y_2) - \rho(x_1, y_1)] + (y_2 - y_1)[q(x_2, y_2) - q(x_1, y_1)] = \tilde{r}(x_2 - x_1)^2 + 2\tilde{s}(x_2 - x_1)(y_2 - y_1) + \tilde{t}(y_2 - y_1)^2 \geq 0. \]

Here \( \tilde{r}, \tilde{s}, \tilde{t} \) stand for the values of \( r, s, t \) in a point of the segment connecting \( z_1 \) with \( z_2 \). Substitute (4) into (5):

\[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \leq (x_2 - x_1)(u_2 - u_1) + (y_2 - y_1)(v_2 - v_1) \]

and hence

\[ |z_2 - z_1| \leq |w_2 - w_1|, \]

equality holding only if \( z_1 = z_2 \) (see H. Lewy \[10\]). Therefore the mapping (4) is one-to-one and it enlarges distances. Denote by \( \Omega \) the image domain of \( C \) under this mapping. On the other hand, going back to the definitions of \( x, y \) and \( \rho, q \) one finds

\[ w = F(\gamma) + iG(\gamma) \equiv W(\gamma). \]

That is to say the domain \( \Omega \) is also the schlicht conformal image of \( \Gamma \) under the mapping function \( W(\gamma) \).

3. It is easy to see that, starting out with a solution \( \phi(x, y) \) of (1), the inverse mapping \( w \rightarrow z \) under all circumstances is harmonic. Furthermore it turns out that the expression \( f = 2\bar{z} - \bar{w} \) which can be regarded as a function of \( u \) and \( v \) is an analytic function of \( w \). The
inequality $|df/dw| < 1$ which is satisfied by its derivative has interesting consequences, see [12].

4. The lemma in question states that there cannot exist a schlicht harmonic mapping of the unit disc $\Gamma$ onto the whole $z$-plane. The proof is obvious since, if $C$ would be the whole $z$-plane then $\Omega$ would have to be the whole $w$-plane. But, at the same time, $\Omega$ is the conformal image of $\Gamma$. This is not possible.

5. Suppose now that $C$, like $\Gamma$, is the unit disc $|z| < 1$. E. Heinz [3] has established an inequality

\[
\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \geq \mu.
\]

Here his constant $\mu$ is independent of the individual harmonic mapping under consideration. Heinz found $\mu \geq 2 - \frac{8}{n} = 2 - \frac{2}{3} = 0.358$. Using the relations derived above one obtains the formula

\[
\frac{r + t}{2 + r + t} \cdot \left| \frac{dW}{d\gamma} \right|^2.
\]

Remembering the properties of the mapping (4) we know that $\Omega$ contains at least a circle of radius 1. Hence, by Schwarz’s lemma, $|dW(0)/d\gamma| \leq 1$. In fact, the sign of equality cannot hold since $\partial(u, v)/\partial(x, y) = 2 + r + t \geq 4$. Furthermore $1/2 \leq (r + t)/(2 + r + t) < 1$. Combining these two inequalities we conclude

\[
\mu \geq 1/2.
\]

6. We wish to mention that H. Hopf has given another simple proof of the value $1/2$ for the constant $\mu$. A similar inequality to (9) holds also for more general univalent mappings, see P. Berg [1], E. Heinz [4; 5]. However, remaining with the harmonic mappings: the best value of $\mu$ is not known. If one takes the polynomial solution $\phi(x, y) = cx^2/2 + y^2/2c$ then $\Omega$ is an ellipse with the semiaxes $1 + c$ and $1 + 1/c$. A computation yields

\[
\lim_{c \to \infty} \frac{r + t}{2 + r + t} = 1, \quad \lim_{c \to \infty} \left| \frac{dW(0)}{d\gamma} \right| = \frac{4}{\pi},
\]

and hence

\[\frac{8}{\pi}\]

\[\mu \geq 0.64,\]

\[\mu \leq 27/2\pi^2.
\]

In a letter of October 26, 1956.

Added in proof: A refinement of the preceding method yields even $\mu \geq 0.64$, as will be shown elsewhere. Therefore, referring to Richert’s example for an upper bound, one knows: $0.64 \leq \mu \leq 27/2\pi^2$. 

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\[
\lim_{c \to \infty} [x_a^2 + x_\beta^2 + y_a^2 + y_\beta^2]_{y=0} = 16/\pi^2.
\]

By an example of H. E. Richert (cf. E. Hopf [6, p. 802]) it is, however, known that the value 16/\pi^2 is too large.

**Literature**


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