ON TRANSITIVE TRANSLATION FUNCTIONS

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From the definition\(^2\) of a fiber space \((E, B, p)\) in terms of a lifting function,

\[ \lambda: \{(e, \omega) \in E \times B' \mid p(e) = \omega(0)\} \rightarrow B' \text{ such that } p \circ \lambda(e, \omega) = \omega, \]

we are led to a translation function

\[ \tau: \{(e, \omega) \mid p(e) = \omega(0)\} \rightarrow E \text{ where } \tau(e, \omega) = \lambda(e, \omega)(1). \]

We may also consider the maps \(\tau(\omega): p^{-1}(\omega(0)) \rightarrow p^{-1}(\omega(1))\) defined by \(\tau(\omega)(e) = \tau(e, \omega)\). A translation function is transitive if \(\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)\) where

\[ \omega_1(1) = \omega_2(0) \text{ and } (\omega_1 \cdot \omega_2)(t) = \begin{cases} \omega_1(2t) & \text{for } 0 \leq t \leq 1/2, \\ \omega_2(2t - 1) & \text{for } 1/2 \leq t \leq 1. \end{cases} \]

The question of when transitive translation functions exist for fiber bundles was raised by W. Hurewicz. The answer this paper supplies is the following.

If a bundle over a finite polyhedron has a structural group \(G\) with no small subgroups then it has a transitive translation function if and only if it is equivalent in \(G\) to an \(H\) bundle where \(H\) is a totally disconnected subgroup of \(G\).

The central result of this paper is that if \(\tau\) is a transitive translation function and the structural group has no small subgroups, then \(\tau(\omega)\) depends only on the homotopy class of \(\omega\).

All spaces we consider will be Hausdorff spaces; path spaces will have the compact-open topology.

Remarks. For path spaces one may take as a basis all sets of the form \(N = \bigcap_{i=1}^{2^n} \left( [i-1/2^n, i/2^n], U_i \right) \).

A sequence \(\omega_n\) converges to the constant \(x_0\) in the path space \(X'\) if and only if every neighborhood \(U\) of \(x_0\) in \(X\) contains all but a finite number of the sets \(\omega_n(t)\). That this is not true for Moore paths pre-

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vents the extension of our results to translation functions defined on Moore paths.

**Definitions.** A triple \((E, B, p)\) is a regular fiber space if there exists a function \(\tau: \{(e, \omega) \mid p(e) = \omega(0)\} \to E\) such that \(p(\tau(e, \omega)) = \omega(1)\) and \(\tau(e, \omega) = e\) if \(\omega(t) = e\). A fiber space is homeomorphic if each of the maps \(\tau(\omega)\) is a homeomorphism.

A fiber space is transitive if \(\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)\).

In the case of a fiber bundle \((E, B, p, Y, G)\) over a paracompact base space \(B\), the Hurewicz uniformization theorem gives the result: \((E, B, p, Y, G)\) is a regular homeomorphic fiber space and \(\tau(\omega)\) may be identified with a member of \(G\) through any two applicable coordinate maps. We shall therefore define a translation function \(\tau\) for the bundle \((E, B, p, Y, G)\) as a regular homeomorphic translation function for \((E, B, p)\) with the property that \(\phi_i^{-1}(\omega(1))\tau(\omega)\phi_j(\omega(0)) \in G\) for some pair (and therefore all pairs) of coordinate functions \(\phi_i, \phi_j\) such that \((\omega(1), \omega(0)) \subseteq V_i \times V_j\).

**Definition.** \(\omega^*\) is a reparametrization of \(\omega\) if there exists a sense preserving homeomorphism \(f\) of the unit interval onto itself such that \(\omega^* = \omega \circ f\).

**Lemma.** If \(\omega^*\) is a reparametrization of \(\omega\) and \(\tau\) a transitive translation function, then \(\tau(\omega) = \tau(\omega^*)\).

From the associativity of the composition of maps we know that \(\tau(\omega_1 \cdot \ldots \cdot \omega_n) = \tau(\omega_n) \circ \ldots \circ \tau(\omega_1)\) is independent of the bracketting of \(\omega_1 \cdot \ldots \cdot \omega_n\). In this proof we shall approximate \(\omega^*\) by a path \(\omega'\) which is obtained from \(\omega\) by letting \(\omega = \omega_1 \cdot \ldots \cdot \omega_2^m\) in the canonical bracketting (defined below) and \(\omega' = \omega_1 \cdot \ldots \cdot \omega_2^m\) in another bracketting. The continuity of \(\tau\) then gives the result. We shall use the phrase "\(\omega = \omega_1 \cdot \ldots \cdot \omega_2^m\) where the bracketting is canonical" to mean \(\omega = \tilde{\omega} \cdot \omega\) where \(\tilde{\omega} = \omega_1 \cdot \ldots \cdot \omega_2^{m-1}\) and \(\omega = \omega_2^{m-1} \cdot \ldots \cdot \omega_2\) where in each case the bracketting is canonical. The canonical bracketting is defined only when the number of factors is a power of 2.

**Proof of Lemma.** Let \(\omega^*(t) = \omega(f(t))\) and \(N\) be a neighborhood of \(\omega^*\). We may assume \(N = \bigcap_{i=1}^{2^n} \left([[(i-1)/2^n, i/2^n] \cap U_i]\right)\). Let \(\mathcal{U}_i\) be a connected neighborhood of \([[(i-1)/2^n, i/2^n] = C_i\) such that \(\omega^*(\mathcal{U}_i) \subseteq U_i\). Choose dyadic rationals \(0 < m_1/2^n < \ldots < m_2^n/2^n = 1\) such that \(m_i/2^n \subseteq f(\mathcal{U}_i) \cap f(\mathcal{U}_{i+1})\). We shall now construct a path \(\omega'\), which will send \(C_i\) into \((m_{i-1}/2^n, m_i/2^n] \subseteq \omega^*(\mathcal{U}_i) \subseteq U_i\).

Let \(\omega = \omega_1 \cdot \ldots \cdot \omega_2^m\) where the bracketting is canonical. Let \(\omega'_i = \omega_{m_{i-1}+1} \cdot \ldots \cdot \omega_{m_i}\) where the bracketting is arbitrary. Define \(\omega' = \omega'_1 \cdot \ldots \cdot \omega'_2^m\) where the bracketting is canonical. Now \(\omega'(C_i) = \omega'_i(I)\)
\[ \omega_{m_{i+1}}(I) = \omega([m_{i+1}/2^m, m_i/2^m]) \subseteq U_i; \] therefore \( \omega' \in N \) which proves the result. q.e.d.

In the remainder of this paper we shall assume that the structural group \( G \) has the following property:

If \( \omega \in G^t \), \( \omega(0) = e \) the identity, then either

1. There exists a sequence \( t_i \) of reals from the unit interval converging to zero and positive integers \( m_i \) such that \( \omega(t_i) \) does not converge to the identity, or

2. There exists a \( t_0 > 0 \) such that \( \omega(t) = e \) for \( t \leq t_0 \).

In particular groups with no small subgroups have this property.

**Theorem.** If \( \tau \) is a transitive lifting function for the bundle \((E, B, p, Y, G)\), then the map \( \tau(\omega) \) depends only on the homotopy class of \( \omega \).

We proceed by a series of lemmas.

**Lemma 1.** If \( \omega_s(t) \) is a homotopy of the loop \( \omega_1(t) \) at \( b_0 \) to the constant loop \( \omega_0(t) = b_0 \), then there exists a number \( s_0 > 0 \) such that \( \tau(\omega_s) = e \), the identity, for \( s \leq s_0 \).

**Proof.** \( \tau(\omega_s) \) is a path in \( G \) such that \( \tau(\omega_0) = e \). Since condition (2) of our lemma, it suffices to show that condition (1) is impossible. Let \( (t_i, m_i) \) be a sequence of reals and positive integers, as in condition (1). By the transitivity of \( \tau \) we have \( \tau(\omega_s) = \tau(\omega_s^{m_i}) \) where \( \omega_s^{m_i} \) is defined by \( \omega_s^{m_i+1} = \omega_s \cdot \omega_s^{m_i} \). Since \( t_i \) converges to zero, we have that \( \omega_s^{m_i} \) converges to the constant path \( \omega_0 \). Since \( \tau \) is continuous, \( \tau(\omega_s^{m_i}) \) converges to the identity; thus (1) cannot hold. q.e.d.

If \( \omega(t) \) is a path, let \( \omega_1(t) \) denote the path defined by \( \omega_1(t) = \omega(1-t) \).

**Lemma 2.** \( (\tau(\omega))^1 = (\tau(\omega^{-1})) \).

**Proof.** Let \( \omega_s(t) = \omega(st) \). Then \( \omega_s \cdot \omega_s^{-1} \) is a homotopy of a loop to the constant. Let \( s_0 = \sup \{ s \in I \mid \tau(\omega_s \cdot \omega_s^{-1}) = e \text{ for } t \leq s \} \). By Lemma 1 \( s_0 \geq 0 \); we shall show by a contradiction that \( s_0 = 1 \).

If \( s_0 < 1 \), then \( \omega_0 \cdot \omega_1 \) is a reparametrization of \( \omega \) where \( \omega_1(t) = \omega(st) \). Consider \( \omega_s(t) = \omega_1(st) \) and \( \tau(\omega_s \cdot \omega_s^{-1}) \). Lemma 1 gives us \( s_1 > 0 \) such that \( \tau(\omega_1 \cdot \omega_1^{-1}) = e \) for \( s < s_1 \). However \( \omega_0 \cdot \omega_1 \) is a reparametrization of \( \omega_0 + r - r s_0 \). Thus

\[
\tau(\omega_0 + r - r s_0 \cdot \omega_0 + r - r s_0) = \tau((\omega_0 \cdot \omega_r) \cdot (\omega_r^{-1} \cdot \omega_0^{-1}))
\]

\[
= \tau(\omega_0^{-1}) \circ \tau(\omega_r^{-1}) \circ \tau(\omega_0) \circ \tau(\omega_0)
\]

\[ = e \text{ the identity for } r \leq s_1. \]

Hence \( \tau(\omega_s^{-1}) = e \) for \( s \leq s_0 + s_1 - s_0 s_1 \). Since \( s_0 \) is maximal we have
\[ s_0 + s_1 - s_0 s_1 \leq s_0 \text{ or } 1 \leq s_0 \text{ which contradicts } s_0 < 1. \text{ q.e.d.} \]

We are now in a position to prove the theorem. The technique is similar to the proof of Cauchy’s theorem. Let \( \Delta \) denote the model two-simplex and \( \Delta_i^n \) the \( i \)th simplex of the \( n \)th barycentric subdivision. Choose loops \( \rho_{n,i} \) which are (clockwise) homeomorphisms of the reals modulo 1 onto the boundary of \( \Delta_i^n \). By the convexity of \( \Delta \) define the path \( \rho_{x,n,i}(t) = (1-t)x + t\rho_{n,i}(0) \) where \( x \in \Delta \). Let \( \partial_x \Delta_i^n \) denote the loop \( \rho_{x,n,i} \cdot (\rho_{n,i}^{-1} \cdot \rho_{x,n,i}) \).

**Proof of Theorem by Contradiction.** Let \( \omega \) be a null homotopic loop such that \( \tau(\omega) \neq e \) the identity. Let \( \sigma: \Delta \to B \) be a singular simplex such that \( \sigma \circ \partial_0 \Delta = \omega \). There must be a simplex \( \Delta'_0 \) of the first barycentric subdivision such that \( \tau(\sigma \circ \partial_0 \Delta'_0) \neq e \) for otherwise \( \tau(\omega) = e \). We continue by induction and find a nested sequence of simplexes \( \Delta_n^n \) such that \( \tau(\sigma \circ \partial_0 \Delta_n^n) \neq e \). Let \( x = \bigcap_{n=0}^\infty \Delta_n^n \). Let \( \rho_x(t) \) be a homotopy of loops at \( x \) such that \( \rho_{1/n} = \partial_x \Delta_n^n \). Then by Lemma 1 there exists \( s_0 > 0 \) such that \( \tau(\sigma \circ \rho_x) = e \) for \( s \leq s_0 \); in particular there is some \( n \) such that \( \tau(\sigma \circ \partial_x \Delta_n^n) = e \). But this implies \( \tau(\sigma \circ \partial_0 \Delta_n^n) = e \) which is the desired contradiction. q.e.d.

**Theorem.** A bundle over a finite polyhedron with a structural group \( G \), which has no small subgroups, has a transitive translation function if and only if it is equivalent in \( G \) to an \( H \) bundle where \( H \) is a totally disconnected subgroup of \( G \).

**Proof.** Let \( \tau \) be the translation function that exists by the Hurewicz uniformization theorem for the \( H \) bundle. Then since the maps \( \tau(\omega_1 \cdot \omega_2) \) and \( \tau(\omega_2) \circ \tau(\omega_1) \) are homotopic and \( H \) is totally disconnected it follows that they are equal. Hence \( \tau \) is transitive.

If, on the other hand, there is a transitive translation function \( \tau \), we can construct a coordinate bundle with a totally disconnected group. Select a point \( b_0 \) in the base space and let \( \phi_0(b_0): Y \to p^{-1}(b_0) \) be one of the coordinate maps restricted to \( Y \times b_0 \). For every vertex \( a_i \) select a contraction of its star to the point \( b_0 \). Thus for every \( x \in \text{St } a_i \) we obtain a path \( \omega_{a_i,x} \) from \( b_0 \) to \( x \). Define the maps \( \phi_{a_i}: Y \times \text{St } a_i \to p^{-1}(\text{St } a_i) \) by \( \phi_{a_i}(y, x) = \tau(\phi_0(y), \omega_{a_i,x}) \). Thus we obtain coordinate maps which are compatible with the original coordinate maps. The subgroup \( H \) of \( G \) which is spanned by the \( g_{a_i,a_j}(x), x \in \text{St } a_i \cap \text{St } a_j \), is the continuous image of the finitely generated fundamental group of the base space and hence \( H \) is totally disconnected. q.e.d.