Corollary 2. If $\phi: H^k(A) \rightarrow H^k(\overline{A})$ is an isomorphism for $k \leq n$ and univalent for $k = n + 1$, and $\phi: H(H^+(X, A)) \rightarrow H(H^+(X, \overline{A}))$ is an isomorphism, then $\phi: H^k(TA) \rightarrow H^k(T\overline{A})$ is also an isomorphism for $k \leq n$ and univalent for $k = n + 1$.

Reference


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ONE-PARAMETER TRANSFORMATION GROUPS
IN THE PLANE

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Very little is known about the action of a one-parameter group $R$ on two-space except when all orbits are circles, in which case the action is completely known [1]. In a forthcoming paper, A. Beck proves that any closed set can act as the set of fixed points for $R$. Hence, a very general description appears to be hopeless. However, here, we are able to prove the following result.

Theorem. Let $E$ be the plane and $R$ the real line acting on $E$ as a group of transformations without fixed points (i.e., no point is left fixed by all of $R$). If $E/R$ is Hausdorff, then $E$ is fibred as a direct product of $R$ and a cross sectioning line. Thus, $R$ is equivalent to a group of translations.

Proof. Let $x \in E$. Since $x$ is not fixed under $R$, there is a closed interval $[-a, a] = T$ about 0 in $R$, and an arc $C \subset E$, $x \in C$ but not an end point of $C$, such that $T^2(C)$ is a compact neighborhood of $x$ and the mapping $(t, c) \mapsto t(c)$ is one to one from $T^2 \times C \rightarrow T^2(C)$. That is, $C$ is a local cross section to the local orbits of $T^2$ [1]. We shall show that $C$ is a local cross section for the orbits of $R$.

Suppose, on the contrary, that for some $z \in C$, there is an $r > a$ such that $r(z) \in T(C)$. Let $b$ be the greatest lower bound of such numbers. Then $b(z) \in -a(C)$, for if not, say $b(z) = t(c)$, $t \in T$, $c \in C$, and $t > -a$, then there is a $t'$, $-a < t' < 0$, such that $t + t' > -a$. Hence $(b + t')(z) = (t + t')(c) \in T(C)$. But $t' < 0$ so that $b + t' < b$. By the choice of $b$, this implies $b + t' < a$. Since this implies $b + t'' < a$ for

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Let $t' \leq t < 0$, $b = a$. That is, for some $t$, $0 < t < a$, $(b + t)(z) \in T(c)$. But this contradicts the fact that $T^2(C)$ is homeomorphic to $T^2 \times C$. Thus $b > a$, and $b(z) \in -a(C)$.

Suppose $b(z) = -a(z)$. Then $[-a, b](z) = R(z)$ is a circle bounding a pre-compact region $A$ by the Jordan curve theorem. Hence, if $z \in A$, $R(z) \subset A$ since orbits cannot intersect. Then, for any $r \in R$, $r(A^-) \subset A^-$. Thus $r$ has a fixed point. For each $n$, let $x_n$ be a fixed point for $1/2^n$. Let $y$ be a limit point of the $x_n$'s. Thus $R(y) = y$, contradicting our hypothesis that $R$ acts without fixed points.

Now we may assume $b(z) \neq -a(z)$. Let $[b(z), -a(z)]$ denote the arc of $-a(C)$ joining $b(z)$ to $-a(z)$. Then $[-a, b](z) \cup [b(z), -a(z)]$ is a simple closed curve in $E$, and thus divides $E$ into two parts $A$ and $B$ one of which is pre-compact. Moreover, if $R_+ = (0, \infty)$, $R_-(\infty, 0)$, $R_+(b(z))$ is contained in one, say it is $A$, while $R_(-a(z))$ is contained in the other, for $R_+(b(z))$ cannot leave $A$ except by crossing $[b(z), -a(z)]$ since an orbit cannot cross itself. But $R_+(b(z))$ cannot leave $T(C)$ except by crossing $a(C)$. A similar argument establishes that $R_-(a(z)) \subset B$.

Moreover, if $z \in A$, and $r > 0$, then $r(z) \in A$ since again $R(z)$ cannot cross $[-a, b](z)$, and cannot leave $T(C)$ through $-a(C)$. Hence, for $r > 0$, $r(A^-) \subset A^\circ$. But $A^\circ$ is a closed two-cell since it is bounded by a simple closed curve. Thus $r$ has a fixed point. Again find $x_n \in A^\circ$ such that $1/2^n(x_n) = x_n$. If $y$ is a limit point of $x_n$, then $R(y) = y$, a contradiction.

This proves that $C$ meets $R(x)$ in exactly one point for each $x \in T(C)$. Thus, $C$ is a local cross section for all of $R$. Since we can find a local cross section for each point of $E$, and since $E/R$ is Hausdorff, $E$ is a fibre bundle over $E/R$ with fibre $R$ and base space a connected one-dimensional manifold. Since $E$ is a principal fibre bundle over $E/R$, and $R$ is the line, there is a cross section $L$ of $E/R$ in $E$ such that the natural mapping $E \times R \to R(E)$ is a homeomorphism onto $E$ [2]. Clearly $L$ must be a line.

Bibliography