THE MEASURE OF THE SET OF ADMISSIBLE LATTICES

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Introduction. Let \( S \) be a Borel set in \( n \)-dimensional space which does not contain the origin 0. We assume that there is no \( X \) so that both \( X \in S \) and \(-X \in S\). We say a point lattice \( \Lambda \) is \( S \)-admissible, if there is no lattice point of \( \Lambda \) in \( S \). We denote by \( A(S) \) the set of \( S \)-admissible lattices and by \( V = V(S) \) the measure of \( S \).

The main result of this paper is

**Theorem 4.** If

\[
V \leq n - 1 \quad \text{and} \quad n \geq 13,
\]

then

\[
m(A(S)) = \int_{\Omega \Lambda_0 \in A(S); \Omega \in F} d\mu(\Omega) = e^{-V(1 - R)},
\]

where

\[
| R | < 6(3/4)^n e^{4V} + V^{n-1}n^{-n+1}e^{V+n}.
\]

Here \( \Omega \) denotes a linear transformation of determinant 1, \( F \) is a fundamental region with respect to the subgroup of unimodular transformations of determinant 1, and \( \mu(\Omega) \) is the invariant measure on the space of linear transformations with determinant 1, defined by C. L. Siegel [5], normalized so that

\[
\int_F d\mu(\Omega) = 1.
\]

\( \Lambda_0 \) denotes the lattice of points with integral coordinates.

Theorem 4 will be used to prove Theorem 5 which is an improvement of the Minkowski-Hlawka Theorem. We also prove two existence theorems which are in a certain sense converses of the Minkowski-Hlawka Theorem (Theorem 6 and Theorem 7).

The author is very indebted to the referee who pointed out some errors of the originally submitted paper and made useful suggestions. The originally stated Theorem 1 was wrong.

Presented to the Society, June 15, 1957; received by the editors November 23, 1956 and, in revised form, June 3, 1957 and October 31, 1957.
1. We define the lattice function

\[ \alpha(\Lambda) = \begin{cases} 
1, & \text{for } \Lambda \in A(S), \\
0, & \text{for } \Lambda \notin A(S), 
\end{cases} \]

and \( \rho(\Lambda) \) to be the number of lattice points of \( \Lambda \) in \( S \). The usual bound for \( \alpha(\Lambda) \), used for the proof of the Minkowski-Hlawka Theorem, is

\[ \alpha(\Lambda) \geq 1 - \rho(\Lambda). \]

In §1 we shall replace (5) by a better bound.

We define for \( 0 \leq j \leq k \leq n, k > 0 \),

\[ p^*(\Lambda) \]

to be the number of \( k \)-tuples \((X_1, \cdots, X_k)\) of different lattice points \( X_i \) of \( \Lambda \) with \( X_1 \in S, \cdots, X_k \in S \) and \( \dim (X_1, \cdots, X_k) = j \). (Here the order is immaterial, that is, we count \( k \) points of a \( k \)-tuple \((X_1, \cdots, X_k)\) only once and not \( k! \) times.)

We further define \( r^*(\Lambda) \) and \( \tau^*(\Lambda) \) by

\[ \tau^*(\Lambda) = \begin{cases} 
\rho^*(\Lambda), & \text{if } k \text{ is even,} \\
\rho^*(\Lambda) + \rho^\alpha(\Lambda), & \text{if } k \text{ is odd,} 
\end{cases} \]

and

\[ \pi^*(\Lambda) = \begin{cases} 
\rho^*(\Lambda), & \text{if } k \text{ is odd,} \\
\rho^*(\Lambda) + \rho^\alpha(\Lambda), & \text{if } k \text{ is even.} 
\end{cases} \]

Since \( 0 \in S \), \( \tau_1(\Lambda) = \rho^1(\Lambda) + \rho^0(\Lambda) = \rho^1(\Lambda) = \rho(\Lambda) \).

The purpose of this section is to prove

**Theorem 1.**

\[ 1 + \sum_{k=1}^{g} (-1)^k \pi_k(\Lambda) \geq \alpha(\Lambda) \geq 1 + \sum_{k=1}^{h} (-1)^k \tau_k(\Lambda), \]

for any odd \( h \leq n \) and any even \( g \leq n \).

For example, we have for \( h = 1 \) and \( h = 3 \)

\[ \alpha(\Lambda) \geq 1 - \rho(\Lambda) \quad \text{and} \quad \alpha(\Lambda) \geq 1 - \rho^1(\Lambda) + \rho^2(\Lambda) - \rho^3(\Lambda) - \rho^2(\Lambda), \]

respectively. For the proof of Theorem 1 we need some lemmas. We consider the numbers

\[ A^h_m = \sum_{k=0}^{h} \binom{m}{k} (-1)^k \quad (0 \leq h \leq m, m > 0). \]
Lemma 1.

\[ A^h_m \leq 0, \text{ if } h \text{ is odd}; \]
\[ A^h_m \geq 0, \text{ if } h \text{ is even}. \]

Proof of Lemma 1. We first assume \( h < m/2 \). Then we have

\[ \left( \begin{array}{c} m \\ r - 1 \end{array} \right) \leq \left( \begin{array}{c} m \\ r \end{array} \right), \]

if \( r \leq h \). Therefore, if \( h \) is odd, we see that

\[ A^h_m = \sum_{r \text{ odd}} \left[ \begin{array}{c} 1 \leq r \leq h \\ r \end{array} \right] \left\{ \left( \begin{array}{c} m \\ r \end{array} \right) - \left( \begin{array}{c} m \\ r - 1 \end{array} \right) \right\} \leq 0; \]

and, if \( h \) is even,

\[ A^h_m = 1 + \sum_{r \text{ even}} \left[ \begin{array}{c} 1 \leq r \leq h \\ r \end{array} \right] \left\{ \left( \begin{array}{c} m \\ r \end{array} \right) - \left( \begin{array}{c} m \\ r - 1 \end{array} \right) \right\} \geq 0. \]

If \( m > h \geq m/2 \), then \( m - (h+1) < m/2 \) and

\[ A^h_m = \sum_{k=0}^h \binom{m}{k} (-1)^k = \sum_{k=0}^m \binom{m}{k} (-1)^k - \sum_{k=h+1}^m \binom{m}{k} (-1)^k \]

\[ = 0 - \sum_{k=0}^{m-(h+1)} \binom{m}{k} (-1)^{m+k} = (-1)^{m+1} A^m_{m-(h+1)}. \]

Thus, if \( h \) is odd, we obtain the following:

If \( m \) is even, then \( A^m_{m-(h+1)} \geq 0 \), \( (-1)^{m+1} = -1 \), and so \( A^h_m \leq 0 \); if \( m \) is odd, then \( A^m_{m-(h+1)} \leq 0 \), \( (-1)^{m+1} = 1 \), and so \( A^h_m \leq 0 \).

In a similar way we can prove that, if \( h \) is even, then \( A^h_m \geq 0 \). If \( m = h \), \( A^m_m = 0 \).

Lemma 2. Let \( a_0, a_1, a_2, \ldots, a_m \) be real non-negative numbers, for which

(7a) \[ 1 = a_0 = a_1, \quad a_{2t} \geq a_{2t+2} \quad (0 \leq 2t \leq m - 2) \]

and

(8a) \[ a_{2t} \leq a_{2t+1} \quad (0 \leq 2t \leq m - 1) \]

hold. Then we have

(9a) \[ \sum_{k=0}^h \binom{m}{k} (-1)^k a_k^2 \leq 0, \]
if either $h$ is odd and $h \leq m$, or if $h = m$.

But if $b_0, b_1, b_2, \ldots, b_m$, are real non-negative numbers, for which

(7b) \[ 1 = b_0 = b_1, \quad b_{2t-1} \geq b_{2t+1} \quad (2 \leq 2t \leq m - 1) \]

and

(8b) \[ b_{2t-1} \leq b_{2t} \quad (2 \leq 2t \leq m) \]

hold, then

(9b) \[ \sum_{k=0}^{g} \binom{m}{k} (-1)^k b_k \geq 0, \]

if either $g$ is even and $g \leq m$, or if $g = m$.

**Proof of Lemma 2.** First we consider the case when (7a) and (8a) hold. We may assume that $a_{2t+1} = a_{2t}$. Then, using partial summation and Lemma 1, we have

\[
\sum_{k=0}^{h} \binom{m}{k} (-1)^k a_k = \sum_{t \text{ odd}} \left[ 1 \leq t \leq h - 1 \right] (a_{t-1} - a_{t+1}) \sum_{k=0}^{t} \binom{m}{k} (-1)^k \\
+ a_h \sum_{k=0}^{h} \binom{m}{k} (-1)^k \leq a_h \sum_{k=0}^{h} \binom{m}{k} (-1)^k.
\]

Now the right side is less than or equal to 0, if $h$ is odd, or if $h = m$. So (9a) is true. Similarly (7b) and (8b) imply (9b).

**Lemma 3.** Let $\Lambda$ be a lattice with $\rho(\Lambda) = m > 0$. We define numbers $a_0, a_1, a_2, \ldots, a_m$ and $b_0, b_1, b_2, \ldots, b_m$ by $a_0 = b_0 = 1$ and

(10) \[ \tau_k(\Lambda) = a_k \binom{m}{k} \quad \text{and} \quad \pi_k(\Lambda) = b_k \binom{m}{k} \quad (1 \leq k \leq m). \]

Now we assert the following: The $a_k$ satisfy (7a) and (8a), the $b_k$ satisfy (7b) and (8b).

**Proof of Lemma 3.** We have

\[ \tau_1(\Lambda) = m = a_1 \binom{m}{1} = a_1 m \]

and therefore $a_1 = 1$. Defining constants $c_k$ by

\[ \rho_k(\Lambda) = c_k \binom{m}{k} \]
we obtain
\[
c_{k+1} \left( \begin{array}{c} m \\ k + 1 \end{array} \right) = \frac{k+1}{\rho_{k+1}(\Lambda)}
\]
\[
= \{ \text{the number of } (k+1)\text{-tuples } (X_1, \ldots, X_{k+1}) \text{ of lattice points of } \\
\Lambda \text{ with } X_1 \in S, \ldots, X_{k+1} \in S \text{ of dimension } k + 1 \}
\]
\[
\leq \frac{k}{\rho_k(\Lambda)} \frac{m-k}{k+1} = c_k \left( \begin{array}{c} m \\ k \end{array} \right) \frac{m-k}{k+1} = c_k \left( \begin{array}{c} m \\ k + 1 \end{array} \right).
\]

The inequality holds because each \((k+1)\)-tuple considered can be represented as the union of a \(k\)-tuple of linearly independent points of \(\Lambda\) in \(S\) and another point of \(\Lambda\) in \(S\) in \(k+1\) ways. But there are \(\rho_k(\Lambda)\) such \(k\)-tuples and a \(k\)-tuple given, there are \(m-k\) other points of \(\Lambda\) in \(S\).

Dividing by
\[
\left( \begin{array}{c} m \\ k + 1 \end{array} \right),
\]
we obtain \(c_{k+1} \leq c_k\). Since, for even \(k > 0\), \(a_k = c_k\), we have \(a_{2t} \geq a_{2t+2}\) for \(t > 0\). Also \(a_0 = a_1 = c_1 \geq c_2 = a_2\). Hence the \(a_k\) satisfy (7a). If \(t > 0\), then
\[
a_{2t+1} \left( \begin{array}{c} m \\ 2t + 1 \end{array} \right) = \tau_{2t+1}(\Lambda) = \frac{2t+1}{\rho_{2t+1}(\Lambda)} + \frac{2t}{\rho_{2t+1}(\Lambda)}
\]
\[
= \{ \text{the number of } (2t+1)\text{-tuples } (X_1, \ldots, X_{2t+1}) \text{ of different lattice points of } \\
\Lambda \text{ satisfying } X_1 \in S, \ldots, X_{2t+1} \in S \text{ of dimension } \geq 2t \}
\]
\[
\geq \frac{2t}{\rho_{2t}(\Lambda)} \frac{m-2t}{2t + 1} = \tau_{2t}(\Lambda) \frac{m-2t}{2t + 1} = a_{2t} \left( \begin{array}{c} m \\ 2t \end{array} \right) \frac{m-2t}{2t + 1} = a_{2t} \left( \begin{array}{c} m \\ 2t + 1 \end{array} \right).
\]

Dividing by
\[
\left( \begin{array}{c} m \\ 2t + 1 \end{array} \right)
\]
we obtain \(a_{2t+1} \geq a_{2t}\) and (8a).

If, in the above proof we replace \(a_k\) by \(b_k\), \(\tau_k\) by \(\pi_k\), even by odd, and in places \(2t+1\) by \(2t\), then we obtain (7b) and (8b).

**Proof of Theorem 1.** Again let \(\Lambda\) be a lattice with \(\rho(\Lambda) = m > 0\). Let the numbers \(a_k\) and \(b_k\) be defined by (10). Then the \(a_k\) satisfy (7a) and (8a), the \(b_k\) satisfy (7b) and (8b). If therefore \(h\) is odd, \(h \leq n\), \(h \leq m\), we have
1 + \sum_{k=1}^{h} (-1)^k \tau_k(\Lambda) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} a_k \leq 0 = \alpha(\Lambda),

by Lemma 2. But if \( h \leq n, h \geq m \), we obtain the same result:

1 + \sum_{k=1}^{h} (-1)^k \pi_k(\Lambda) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} b_k \geq 0 = \alpha(\Lambda).

In case \( g \) is even, \( g \leq n, g \geq m \), we have

1 + \sum_{k=1}^{g} (-1)^k \pi_k(\Lambda) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} b_k \geq 0 = \alpha(\Lambda);

and for \( g \leq n, g \geq m \)

1 + \sum_{k=1}^{g} (-1)^k \pi_k(\Lambda) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} b_k \geq 0 = \alpha(\Lambda).

Therefore Theorem 1 is true if \( \rho(\Lambda) > 0 \). It is evidently true if \( \rho(\Lambda) = 0 \).

2. We now calculate the integrals of \( \rho_k(\Lambda) \) and \( \rho_k^{-1}(\Lambda) \) over the space of lattices with determinant 1.

**Theorem 2.** Suppose \( k < n \). Then \( \rho_k(\Lambda) \) is Borel-measurable in the space of lattices of determinant 1 and

(11) \[ R_k^* = \int_{F} \rho_k(\Omega\Lambda_0) d\mu(\Omega) = \frac{1}{k!} V^k. \]

**Proof of Theorem 2.** First, by the definition of \( \rho_k(\Lambda) \), we see

(12) \[ \rho_k(\Lambda) = \frac{1}{k!} \sum \left[ X_1 \in \Lambda, \ldots, X_k \in \Lambda \atop \dim (X_1, \ldots, X_k) = j \right] \rho(X_1) \cdots \rho(X_k), \]

where \( \rho(X) \) is the characteristic function of \( S \).

On the other hand, we observe the following theorem, stated by C. L. Siegel [5] and proved by C. A. Rogers \[2\]: If

\[ \psi(\Lambda) = \sum \left[ X_1 \in \Lambda, \ldots, X_k \in \Lambda \atop \dim (X_1, \ldots, X_k) = k \right] \rho(X_1) \cdots \rho(X_k), \]

then

\[ \int_{F} \psi(\Omega\Lambda_0) d\mu(\Omega) \]

\[ C. A. Rogers \[2\], Theorem 3, take \( h = 0. \)
exists and is equal to
\[ \int \cdots \int \rho(X_1) \cdots \rho(X_k) dX_1 \cdots dX_k. \]

Theorem 2 is an immediate consequence of these two results.

**Theorem 3.** Suppose \( k < n \). Then \( \rho_k^{k-1}(\Lambda) \) is Borel measurable in the space of lattices with determinant 1, and
\[ R_k^{k-1} = \int \rho_k^{k-1}(\Omega \Lambda_0) d\mu(\Omega) \]
\[ = \frac{1}{k!} \sum_{i=1}^{k} \sum_{q=1}^{\infty} \sum_{D} \frac{1}{q^n} \int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho\left( \sum_{i=1}^{k-1} \frac{d_i}{q} X_i \right) dX_1 \cdots dX_{k-1}. \]

Moreover,
\[ R_k^{k-1} \leq \frac{V^{k-1}}{(k-1)!} \left[ 3^{k(3/4)^{n/2}} + 5^k 2^{-n} \right]. \]

The sum in (13) is over all integral vectors \( D = (d_1, \cdots, d_{k-1}) \), which have highest common factor relative prime to \( q \), and which obey \( |d_j| < q \) for \( j < l \) and \( |d_j| \leq q \) for \( j \geq l \). Further, if \( q = 1 \), \( D \) is not \((0, 0, \cdots, 0)\) nor of the form \((0, \cdots, 0, 1, 0, \cdots, 0)\).

Before we can give a proof of Theorem 3 we need some lemmas.

**Lemma 4.**
\[ \sum_{X_1 \in \Lambda, \cdots, X_k \in \Lambda} \dim (X_1, \cdots, X_k) = k - 1 \quad \rho(X_1) \cdots \rho(X_k) \quad X_i \neq X_j \text{ if } i \neq j \]
\[ = \sum_{l=1}^{k} \sum_{q=1}^{\infty} \sum_{D} \sum_{Y_1 \in \Lambda, \cdots, Y_{k-1} \in \Lambda} \dim (Y_1, \cdots, Y_{k-1}) = k - 1 \quad \rho(Y_1) \cdots \rho(Y_{k-1}) \rho\left( \sum_{i=1}^{k-1} \frac{d_i}{q} Y_i \right), \]

where the sum on the right hand side is to be taken over the same set of vectors \( D \) as in Theorem 3.
Proof of Lemma 4. If \( X_1, \ldots, X_k \) is in the sum of the left hand side of (15), then \( \dim (X_1, \ldots, X_k) = k - 1 \). Hence, the vectors \( X_1, \ldots, X_k \) span a \((k - 1)\)-dimensional space. In this space we construct a system of orthogonal unit vectors \( e_1, e_2, \ldots, e_{k-1} \). We write \( X_j \) in the form

\[
X_j = \sum_{i=1}^{k-1} a_{ij} e_i \quad (1 \leq j \leq k).
\]

We define \( A_j \) \((1 \leq j \leq k)\) to be the determinant

\[
\begin{vmatrix}
a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{k-1,1} & \cdots & a_{k-1,j-1} & a_{k-1,j+1} & \cdots & a_{k-1,k}
\end{vmatrix}.
\]

There exists a unique \( l \), such that

\[
|A_j| < |A_l| \text{, if } j < l, \quad \text{and} \quad |A_j| \leq |A_l|, \text{ if } j \geq l.
\]

This \( k \)-tuple \((X_1, \ldots, X_k)\) corresponds to the \((k - 1)\)-tuple \((Y_1, \ldots, Y_{k-1})\), defined by

\[
Y_1 = X_1, \ldots, Y_{l-1} = X_{l-1}, \\
Y_l = X_{l+1}, \ldots, Y_{k-1} = X_k,
\]

and to the number \( l \), to the vector \( D = (d_1, \ldots, d_{k-1}) \) and \( q \), uniquely determined by

\[
X_l = \sum_{i=1}^{k-1} \frac{d_i}{q} Y_i
\]

and

\[
g.c.d. (d_1, \ldots, d_{k-1}, q) = 1.
\]

Because of our choice of \( l \) to make \( |A_l| \) maximal we have

\[
|d_t| < q, \text{ if } t < l, \quad \text{and} \quad |d_t| \leq q, \text{ if } t \geq l.
\]

If \( q = 1 \), then \( D(d_1, \ldots, d_{k-1}) \) is not of the form \((0, 0, \ldots, 0)\) or \((0, \ldots, 0, 1, 0, \ldots, 0)\).

Since \( l, d, q, Y_j \) do not depend on any particular choice of the unit vectors \( e_1, \ldots, e_{k-1} \), there corresponds to each term on the left side of (15) exactly one term on the right hand side. If, conversely, there are \( l, D, q, Y_j \) on the right side of (15), then we take the correspondence.
\[ X_1 = Y_1, \ldots, X_{l-1} = Y_{l-1}, \quad X_l = \sum_{i=1}^{k-1} \frac{d_i}{q} Y_i, \]
\[ X_{l+1} = Y_l, \ldots, X_k = Y_{k-1}. \]

These two mappings are one-one and inverse to each other. This proves the lemma.

**Lemma 5 (C. A. Rogers).** Let \( \rho(X_1, \ldots, X_m) \) be a Borel measurable function which is integrable in the Lebesgue sense over the whole \((X_1, \ldots, X_m)\)-space. Let \( q \) be a positive integer and \( D = (d_1, \ldots, d_m) \) be an integral vector with highest common factor relatively prime to \( q \). Then the lattice function

\[ \omega(\Lambda) = \sum_\text{dim } (X_1, \ldots, X_m) = m \]
\[ \sum_{i=1}^{n} \frac{d_i}{q}, \quad X_i \in \Lambda \]

is Borel measurable in the space of lattices of determinant 1, and

\[ \int \omega(\Omega_0) \, d\mu(\Omega) = \frac{1}{q^n} \int \cdots \int \rho(X_1, \ldots, X_m) \, dX_1 \cdots dX_m. \]

**Proof of Lemma 5.** Lemma 5 is essentially the case \( h = 1 \) of Theorem 3 of C. A. Rogers [2]. The only difference is that we write \( l/q \) instead of \( \epsilon^i/q \) as in Rogers, where \( \epsilon_i = \text{g.c.d.}(\epsilon_i, q) \) and \( \epsilon_i \) is the elementary divisor of the matrix \( D \). But since g.c.d. \((d_1, \ldots, d_m, q) = 1\), we have \( \epsilon_i = \text{g.c.d.}(\epsilon_i, q) = 1 \).

**Lemma 6 (C. A. Rogers).** If \( \rho(X) \) is a characteristic function, then

\[ \int \int \rho(X) \rho(Y) \rho(X + Y + a) \, dX \, dY \leq 2(3/4)^{n/2} \left( \int \rho(X) \, dX \right)^2. \]

**Proof of Lemma 6.** See C. A. Rogers [3, Lemma 5].

**Proof of Theorem 3.** (13) is a straightforward consequence of (12), Lemma 4 and Lemma 5 (take \( m = k - 1 \)). Therefore only (14) remains to be proved. (14) implies that both sides of (13) are finite. We evidently have

\[ R_k^{k-1} \leq \frac{1}{k!} k \sum_{q=1}^\infty \sum_D \frac{1}{q^n} \int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho \left( \sum_{i=1}^{k-1} \frac{d_i}{q} X_i \right) \, dX_1 \cdots dX_{k-1}, \]
but now the summation is to be taken over all integral $D$ with highest common factor relatively prime to $q$ and $|d_j| \leq q$. If $q = 1$, then $D \neq (0, 0, \cdots, 0)$ and $D \neq (0, \cdots, 0, 1, 0, \cdots, 0)$.

In (19) we mean that the inequality holds, if the right hand side is finite. We estimate the sum on the right hand side. We derive upper bounds (A) for the terms with $q = 1$ and (B) for terms with $q > 1$.

(A) There are $\leq 3^{k-1}$ possibilities for $D$. $D$ either has two elements $d_i, d_j$, both different from zero, or $D$ is the form $(0, \cdots, 0, -1, 0, \cdots, 0)$. In the first case we have, by Lemma 6,

$$\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho(\pm X_{i_1} \pm X_{i_2} \pm \cdots) dX_1 \cdots dX_{k-1}$$

$$\leq 2(3/4)^{n/2} \left( \int \rho(X) dX \right)^{k-1} = 2(3/4)^{n/2} V^{k-1}.$$ 

If $D$ is of the form $(0, \cdots, 0, -1, 0, \cdots, 0)$, then

$$\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho(-X_i) dX_1 \cdots dX_{k-1} = 0.$$ 

Thus

$$\sum_D \frac{1}{1^n} \int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho \left( \frac{\sum_{i=1}^{k-1} d_i}{q} X_i \right) dX_1 \cdots dX_{k-1}$$

$$\leq \left[ 3(3/4)^{n/2} \right] V^{k-1}.$$ 

(B) For a fixed $q > 1$ the number of vectors $D$ is at most $(2q + 1)^{k-1} \leq (5/2)^{k-1} q^{k-1}$. Consequently,

$$\sum_{q=2}^{\infty} \sum_D \frac{1}{q^n} \int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho \left( \frac{\sum_{i=1}^{k-1} d_i}{q} X_i \right) dX_1 \cdots dX_{k-1}$$

$$\leq (5/2)^{k-1} \sum_{q=2}^{\infty} q^{k-1-n} V^{k-1} \leq (5/2)^{k-1} 2^{k-1-n} \sum_{q=2}^{\infty} \frac{1}{q^2} V^{k-1}$$

$$< (5/2)^{k-1} 2^{k-1-n} V^{k-1} < 5^{k-2} V^{k-1}.$$ 

By (19), (20) and (21) we get the upper bound

$$K_k^{k-1} \leq \frac{1}{(k-1)!} \left[ 3(3/4)^{n/2} + 5^{k-2} n \right] V^{k-1}.$$ 

3. **Proof of Theorem 4.** Assume that (1) is satisfied. If $h$ is odd and $h < n$, we infer from Theorem 1 that

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\[
\int_F \alpha(\Omega\Lambda_0) d\mu(\Omega) \geq 1 + \sum_{k=1}^h (-1)^k \int_F \tau_k(\Omega\Lambda_0) d\mu(\Omega)
\]
\[
\geq 1 + \sum_{k=1}^h (-1)^k R_k - \sum_{k=2}^h R_k^{k-1}
\]
\[
\geq 1 + \sum_{k=1}^h (-1)^k \frac{V^k}{k!} - \sum_{k=2}^h \left[3^k(3/4)^{n/2} + 5^k2^{-n}\right] \frac{V^{k-1}}{(k-1)!}.
\]

Using the Taylor expansion of \(e^{-v}\) with a remainder after \(h+1\) terms, we see that this implies that

\[
\int_F \alpha(\Omega\Lambda_0) d\mu(\Omega)
\]
\[
\geq e^{-V} - \sum_{k=2}^h \left[3^k(3/4)^{n/2} + 5^k2^{-n}\right] \frac{V^{k-1}}{(k-1)!} - \frac{V^{h+1}}{(h+1)!}.
\]

If \(g\) is even and \(g < n\), we obtain, in a similar way

\[
\int_F \alpha(\Omega\Lambda_0) d\mu(\Omega) \leq e^{-V} + \sum_{k=2}^g \left[3^k(3/4)^{n/2} + 5^k2^{-n}\right] \frac{V^{k-1}}{(k-1)!} + \frac{V^{g+1}}{(g+1)!}.
\]

A combination of both these inequalities gives

(2) \[ m(A(S)) = \int_F \alpha(\Omega\Lambda_0) d\mu(\Omega) = e^{-V}(1 - R), \]

and

(22) \[ -e^V \left[ \sum_{k=1}^g \frac{V^{k-1}}{(k-1)!} \right] \leq R \]

\[
\leq e^V \left[ \sum_{k=2}^h \left[3^k(3/4)^{n/2} + 5^k2^{-n}\right] \frac{V^{k-1}}{(k-1)!} + \frac{V^{h+1}}{(h+1)!} \right].
\]

But, provided \(1 \leq k \leq n\), we have

(23) \[ 5^k2^{-n} = 3^k(5/3)^k2^{-n} < 3^k(5/6)^n < 3^k(3/4)^{n/2}. \]

So

(24) \[ \sum_{k=2}^h \left[3^k(3/4)^{n/2} + 5^k2^{-n}\right] \frac{V^{k-1}}{(k-1)!} e^V < 6(3/4)^{n/2} \sum_{k=2}^h \frac{3^{k-1} V^{k-1}}{(k-1)!} e^V < 6(3/4)^{n/2} e^V. \]
Now take $h$ to be odd and to have either the value $n - 1$ or the value $n - 2$. Then as $V < n - 1$ we have

$$
\frac{V^{h+1}}{(h+1)!} e^V \leq \frac{V^{n-1}}{(n-1)!} e^V.
$$

Since

$$
e^n > \frac{n^{n-1}}{(n-1)!},
$$

it follows that

$$
(25) \quad \frac{V^{h+1}}{(h+1)!} e^V < V^{n-1} n^{n+1} e^{V+n}.
$$

Using (24) and (25) in (22) we obtain

$$
R < 6(3/4)^{n/2} e^{4V} + V^{n-1} n^{n+1} e^{V+n}.
$$

A similar argument shows that

$$
R > -6(3/4)^{n/2} e^{4V} - V^{n-1} n^{n+1} e^{V+n}.
$$

A combination of these inequalities gives (3) and proves Theorem 4.

**Theorem 5 (Improvement of the Minkowski-Hlawka Theorem).** Let $S$ be a Borel set, not containing the origin 0. Suppose

$$
(26) \quad V \leq \frac{1}{8} n \log 4/3 - \frac{1}{2} \log 3.
$$

Then there exists an admissible lattice $\Lambda$ with determinant 1.

In the original Minkowski-Hlawka Theorem there is $V < 1$ instead of (26). It was first proved by E. Hlawka [1]. In the meantime it was proved to be true for $V < 2/(1+2^{1-n})(1+3^{1-n})$ by the author [4] and for $V \leq n^{1/2}/6$ if $n$ is sufficiently large by C. A. Rogers [3].

**Proof of Theorem 5.** We may assume that $X \in S$ implies $-X \not\in S$. We may also assume $n \geq 13$, because if $n < 13$, then (26) yields $V < 1$, and the theorem is true. (26) implies (1). Hence (2) and (3) hold. (26) also implies

$$
6(3/4)^{n/2} e^{4V} \leq 2/3.
$$

Further, as $\log 4/3 < 1/3$, we have $V < n/24$. Also $e^{25/24} < 24/23$. Thus

$$
V^{n-1} n^{n+1} e^{V+n} < (1/24)^{n-1} e^{25n/24} < 24(24)^{-n}(24/23) = 24(23)^{-n} < 1/3.
$$
Combining these we obtain $|R| < 1$, so that $m(A(S)) > 0$. Consequently, there exists an admissible lattice of determinant 1.

**Theorem 6.** Let $S$, $T$ be two Borel sets. Assume that $X \in T$ yields $-X \notin S \cup T$ and that $0 \in S$. Further assume

\[ V(S) \leq \frac{1}{16} n \log 4/3 - \frac{1}{2} \log 3 - 4(3/4)^{n/2}, \]

or

\[ V(S \cup T) \geq V(S) + 4(3/4)^{n/4}. \]

Then there exists a lattice $\Lambda$ with determinant 1 which is $S$-admissible, but not $T$-admissible.

**Proof of Theorem 6.** We may assume that $X \in S$ yields $-X \notin S$. Then never both $X \in S \cup T$ and $-X \notin S \cup T$. We introduce $S_1 = S$, $S_2 = S \cup T$. We may assume that equality holds in the second equation (27), that is,

\[ V(S_2) = V(S_1) + 4(3/4)^{n/4}. \]

Then

\[ V(S_i) \leq \frac{1}{16} n \log 4/3 - \frac{1}{2} \log 3. \]

Writing $\alpha_j(\Lambda) = \alpha_{S_j}(\Lambda)$, $V_j = V(S_j)$, $R_j = R(S_j)$, $c = (3/4)^{n/4}$, and applying Theorem 4 we infer

\[ \int \alpha_i(\Omega \Lambda_0) d\mu(\Omega) = e^{-V_i(1 - R_i)}, \]

where

\[ |R_i| \leq \frac{2}{3} (3/4)^{n/4} + 24(23)^{-n} \leq (3/4)^{n/4} = c < \frac{1}{2}. \]

Hence

\[ \int \left[ \alpha_1(\Omega \Lambda_0) - \alpha_2(\Omega \Lambda_0) \right] d\mu(\Omega) = e^{-V_1(1 - R_1)} - e^{-V_2(1 - R_2)} \]

\[ = e^{-V_2}[e^{V_2 - V_1}(1 - R_1) - (1 - R_2)] \geq e^{-V_2}[e^{4c}(1 - c) - (1 + c)] \]

\[ > e^{-V_2}[4c(1 - c) - (1 + c)] = e^{-V_2}(2c - 4c^2) > 0. \]

Consequently, there exists a lattice $\Lambda$ satisfying $\alpha_1(\Lambda) - \alpha_2(\Lambda) > 0$. This implies $\alpha_1(\Lambda) = 1$, $\alpha_2(\Lambda) = 0$. Therefore there is a point of $\Lambda$ in $S_2 = S \cup T$, but no point of $\Lambda$ in $S_1 = S$. Thus $\Lambda$ is $S$-admissible, but not $T$-admissible.
Theorem 7. Let $S_1, \ldots, S_m$ be $m$ Borel sets in $R_n$, $n \geq 13$, each so that $X \in S$ yields $-X \in S$ and with

$$\sum_{j=1}^{m} e^{-W_j} \left[1 + R(n, V_j)\right] \leq 1,$$

where $W_j = \min \left(V_j, n-1\right)$ and $R(n, V) = 6(3/4)^{n/2}e^{4V} + V^{n-1}n^{-n+1}e^{V+n}$. Then there exists a lattice with determinant 1 which has at least one point in each $S_j$.

Proof of Theorem 7. Clearly it is enough to prove the theorem if $V_j \leq n-1$. We obtain

$$\int \left[ \sum_{j=1}^{m} \alpha_j(\Omega \Lambda_0) \right] d\mu(\Omega) < \sum_{j=1}^{m} e^{-V_j} \left[1 + R(n, V_j)\right] \leq 1.$$

Consequently, there exists a lattice $\Lambda$ such that $\sum_{j=1}^{m} \alpha_j(\Lambda) = 0$ and $\Lambda$ is not admissible for any $S_j$.

References


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