THE GROUP OF ORDER PRESERVING AUTOMORPHISMS
OF AN ORDERED ABELIAN GROUP

PAUL CONRAD

In this note we shall use the terminology and notation from [2, pp. 516–517]. In particular, order will always mean linear order. Some answers are given to the question: When can the group \( \mathfrak{A} \) of \( o \)-automorphisms of an abelian \( o \)-group \( G \) be ordered? We prove that \( \mathfrak{A} \) can be ordered provided that the rank of \( G \) is well ordered and the \( o \)-automorphism group of each component of \( G \) is isomorphic to a subgroup of the positive rationals. Thus every torsion free abelian group \( G \) admits an ordering for which the corresponding \( \mathfrak{A} \) can be ordered. Our results provide examples of nonabelian \( o \)-groups, and they can be used to obtain information about the underlying group \( G \).

1. Representation of an automorphism group by a group of matrices. Whenever possible we shall represent automorphisms by matrices. This is a straightforward generalization of the classical case (for example, see Kurosh [3, p. 156]), and no proofs will be given. Let \( \Gamma \) be an ordered set, and for each \( \gamma \in \Gamma \), let \( D_{\gamma} \) be an abelian group. We use the same symbol 0 to denote the identity element of each of the \( D_{\gamma} \). Let \( E_{\gamma} \) be the ring of endomorphisms of \( D_{\gamma} \), and let \( H(D_{\alpha}, D_{\beta}) \), where \( \alpha, \beta \in \Gamma \) and \( \alpha \neq \beta \), be the group of homomorphisms of \( D_{\alpha} \) into \( D_{\beta} \). Finally let \( G \) be the restricted direct sum of the \( D_{\gamma} \), and let \( \Delta \) be the set of all square matrices \( [\pi_{\alpha \beta}] \), where \( \alpha, \beta \in \Gamma \), \( \pi_{\alpha \alpha} \in E_{\alpha} \), and \( \pi_{\alpha \beta} \in H(D_{\alpha}, D_{\beta}) \) for \( \alpha \neq \beta \), that satisfy:

(*) For \( \alpha \) fixed and each \( d \in D_{\alpha} \), \( d \pi_{\alpha \beta} \neq 0 \) for at most a finite number of the \( \beta \).

If \( (\cdots, d_{\alpha}, \cdots) \in G \) and \( [\pi_{\alpha \beta}] \in \Delta \), then

\[
(\cdots, d_{\alpha}, \cdots)[\pi_{\alpha \beta}] = (\cdots, \sum_{\alpha \in \Gamma} d_{\alpha} \pi_{\alpha \beta}, \cdots) \in G.
\]

It is easy to verify that this mapping is an endomorphism of \( G \). Conversely if \( \pi \) is an endomorphism of \( G \), then \((0, \cdots, 0, a_{\alpha}, 0, \cdots, 0)\pi = (\cdots, b_{\beta}, \cdots)\) where at most a finite number of the \( b_{\beta} \) are non-zero. Define \( a_{\alpha} \pi_{\alpha \beta} = b_{\beta} \), then we have a mapping of \( \pi \) upon \( [\pi_{\alpha \beta}] \in \Delta \).

This mapping is an isomorphism of the ring of all endomorphisms of \( G \) onto \( \Delta \). Suppose that each \( D_{\gamma} \) is an \( o \)-group, and define \((\cdots, d_{\gamma}, \cdots) \in G \) positive if it is not zero and the nonzero com-
ponent $d_y$ with greatest subscript $\gamma$ is positive. Then $G$ is an abelian $o$-group, and for each $\gamma \in \Gamma$, $C_\gamma = \{ (\cdots, g_\alpha, \cdots) \in G : g_\alpha = 0 \text{ for all } \alpha > \gamma \}$ is a convex subgroup of $G$. Let $\mathfrak{A}$ be the group of all $o$-automorphisms of $G$, and for each $\gamma \in \Gamma$, let $\mathfrak{D}_\gamma$ be the group of all $o$-automorphisms of $D_\gamma$. Let $T = T(\Gamma, D_\gamma)$ be the multiplicative semigroup of all the triangular matrices $[\pi_{ab}] \in \Delta$ that satisfy:

(a) $\pi_{\gamma\gamma} \in \mathfrak{D}_\gamma$ for all $\gamma \in \Gamma$.

(b) If $\alpha, \beta \in \Gamma$, $\beta > \alpha$ and $d \in D_\alpha$, then $d\pi_{a\beta} = 0$.

$[\pi_{ab}] \in T$ corresponds to an $o$-isomorphism of $G$ into itself, and $[\pi_{ab}]$ corresponds to an $o$-automorphism of $G$ if and only if $[\pi_{ab}]$ is a unit in $T$. Let $U = U(\Gamma, D_\gamma)$ be the group of units of $T$.

**Lemma 1.** If each of the convex subgroups $C_\gamma$ of $G$ is invariant with respect to $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to $U$.

**Proof.** By the isomorphism defined in the preceding discussion, $\mathfrak{A}$ is mapped onto a subgroup of the group of all matrices in $\Delta$ that have inverses in $\Delta$. Let $\pi \in \mathfrak{A}$ and let $[\pi_{ab}]$ be the corresponding matrix. For each $\gamma \in \Gamma$, $C_\gamma [\pi_{ab}] = C_\gamma$. It follows that $[\pi_{ab}]$ satisfies (b). Then since $\pi$ is an $o$-automorphism, $[\pi_{ab}]$ satisfies (a).

**Corollary.** If each $D_\gamma$ is a subgroup of the reals (naturally ordered), and the only $o$-permutation of $\Gamma$ is the identity permutation, then $\mathfrak{A}$ is isomorphic to $U$.

For under these restrictions it is easy to verify that every convex subgroup of $G$ is invariant with respect to $\mathfrak{A}$.

**Remark.** If each $D_\gamma$ is the group of reals, then $\mathfrak{A}$ is isomorphic to $U$ if and only if the only $o$-permutation of $\Gamma$ is the identity permutation.

A subgroup $S$ of $G$ is a $c$-subgroup if for every $\gamma \in \Gamma$ and $d \in D_\gamma$ there exists an element $(\cdots, g_\alpha, \cdots) \in S$ such that $g_\gamma = d$ and $g_\alpha = 0$ for all $\alpha > \gamma$.

**Lemma 2.** Suppose that each $D_\gamma$ is a nonzero $d$-closed subgroup of the reals. Then $U = T$ if and only if $\Gamma$ is well ordered.

**Proof.** Each $[\pi_{ab}] \in T$ induces an $o$-isomorphism of $G$ onto a $c$-subgroup of $G$. If $\Gamma$ is well ordered, then $G$ has no proper $c$-subgroups [1, p. 8]. Thus $G[\pi_{ab}] = G$ for all $[\pi_{ab}] \in T$, hence $U = T$.

Conversely assume that $\Gamma$ is not well ordered. Then we can find a set $\{h_{\gamma_i}\}_{i=1}^\infty$ of nonzero elements such that $h_{\gamma_i} \in D_{\gamma_i}$ and $\gamma_1 > \gamma_2 > \gamma_3 > \cdots$. Let $h_{\gamma_i}^*$ be the smallest $d$-closed subgroup of $D_{\gamma_i}$ that contains $h_{\gamma_i}$, and let $B$ be the restricted direct sum of the $h_{\gamma_i}^*$. Define $h_{\gamma_i}^* = h_{\gamma_{i+1}} + h_{\gamma_i}$. Since $B$ is a vector space over the rationals
and the $h_{\gamma_i}$ form a basis for $B$, there is a unique extension of $\pi$ to an $o$-isomorphism $\sigma$ of $B$ into $B$. Note that $(0, \cdots, 0, h_{\gamma_i}, 0, \cdots, 0)$ does not belong to $B\sigma$. For each $\gamma_i$, let $K_{\gamma_i}$ be a subgroup of $D_{\gamma_i}$ such that $D_{\gamma_i} = h_{\gamma_i}^* \oplus K_{\gamma_i}$. This is possible since the $h_{\gamma_i}^*$ are d-closed. Let $E$ be the restricted direct sum of the $K_{\gamma_i}$ for $i = 1, 2, \cdots$ and the $D_\alpha$ for all $\alpha \neq \gamma_1, \gamma_2, \cdots$. Then $G = B \oplus E$. For each $g = b + e$ in $G$ let $g\tau = b\sigma + e$. It is easy to verify that $\tau$ is an $o$-isomorphism of $G$ onto a proper subgroup of $G$, and that the corresponding matrix $[\pi_{\alpha\beta}]$ belongs to $o T$. Thus $U \neq T$.

**Corollary.** If each $D_{\gamma}$ is a subgroup of the reals and $\Gamma$ is well ordered, then $\mathcal{A}$ is isomorphic to $U$ and $U = T$.

2. **The group $\mathcal{B}$.** For the rest of this paper $G$ will denote an abelian $o$-group, and $\Gamma$ will denote the set of all pairs of convex subgroups $G_\gamma, G_\alpha$ of $G$ such that $G_\gamma$ covers $G_\alpha$.

**Lemma 3.** If $G^*$ is the d-closure of $G$ and $\pi$ is an $o$-automorphism of $G$, then there exists a unique $o$-automorphism $\pi^*$ of $G^*$ such that $g\pi^* = g\pi$ for all $g \in G$. Moreover the mapping $\pi \to \pi^*$ is an isomorphism of the group $\mathcal{A}$ of all $o$-automorphisms of $G$ into the group $\mathcal{A}^*$ of $o$-automorphisms of $G^*$. Thus $\mathcal{A}$ can be ordered if $\mathcal{A}^*$ can be ordered.

**Proof.** For each $g^* \in G^*$ there exists a positive integer $n$ such that $ng^* \in G$. Let $g^*\pi^*$ be the solution of $nx = (ng^*)\pi$. The verification that $\pi^* \in \mathcal{A}^*$ and that the mapping $\pi \to \pi^*$ is an isomorphism is not difficult.

For the remainder of this section we shall assume that $G$ is d-closed. Let $\mathcal{B} = \{\pi \in \mathcal{A}: (G_\gamma + g)\pi = G_\gamma + g$ for all $\gamma \in \Gamma$ and all $g \in G_\gamma\}$. That is, $\mathcal{B}$ consists of all those $o$-automorphisms of $G$ that induce the identity automorphism on each of the components $G_\gamma/G_\alpha$ of $G$. $\mathcal{B}$ is a normal subgroup of $\mathcal{A}$. For let $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $a \in G_\gamma$. $G_\alpha \alpha$ and $G_\gamma \alpha$ are convex subgroups and $G_\alpha \alpha$ covers $G_\alpha$. Therefore $(G_\gamma + a)\beta \alpha^{-1} = (G_\gamma + a\alpha)\beta \alpha^{-1} = (G_\alpha + a\alpha)\alpha^{-1} = G_\gamma + a$. Since $G$ is d-closed, $G_\gamma$ and $G_\alpha$ are d-closed for all $\gamma \in \Gamma$. Thus $G = G_\gamma \oplus D_\gamma$, where $D_\gamma$ is a d-closed subgroup of $G$. Therefore $D_\gamma$ is $o$-isomorphic to a subgroup of the reals. Let $\mathcal{B}_\gamma(\mathcal{B}_\gamma)$ denote the group of all $o$-automorphisms of $G_\gamma$ that induce the identity automorphism on all components. By Lemma 1, $\mathcal{B}_\gamma$ is isomorphic to the multiplicative group of all matrices of the form

$$[a] = \begin{bmatrix} a_{11} & \theta \\ a_{21} & 1 \end{bmatrix}.$$
where $a_{11} \in \mathfrak{B}_{\gamma}$ and $a_{21} \in H(D_{\gamma}, G_{\gamma})$. Here and for the rest of this note 1 will denote an identity automorphism and $\theta$ a zero homomorphism. It will be clear what group 1 refers to ($D_{\gamma}$ in this case), and what group $\theta$ refers to ($H(G_{\gamma}, D_{\gamma})$ in this case). In order to show that an ordering of $\mathfrak{B}_{\gamma}$ can be extended to an ordering of $\mathfrak{B}^\gamma$ we need the following lemma.

**Lemma 4.** If $A$ is an abelian $o$-group, then $H(G, A)$ can be ordered so that if $h \in H$ is positive and $\alpha$ is an $o$-automorphism of $A$, then $h \alpha$ is positive.

**Proof.** Since $G$ is torsion free and $d$-closed, it is a vector space over the rationals $\mathbb{Q}$. Choose and well order a basis $g_1, g_2, \cdots$ for $G$. Define $h \in H$ positive if $h \neq 0$ and $g_i h > 0$ in $A$, where $g_i$ is the first basis element for which $g_i h \neq 0$.

**Corollary.** $\mathfrak{B}_{\gamma}$ is isomorphic to a subgroup of $\mathfrak{B}^\gamma$ and any ordering of $\mathfrak{B}_{\gamma}$ can be extended to an ordering of $\mathfrak{B}^\gamma$.

**Proof.** Order $H(D_{\gamma}, G_{\gamma})$ so that if $h \in H$ is positive and $\pi \in \mathfrak{B}_{\gamma}$, then $h \pi$ is positive. Define $[a]$ positive if $a_{11} > 1$ or $a_{11} = 1$ and $a_{21} > \theta$. It is easy to verify that this definition orders $\mathfrak{B}^\gamma$.

**Theorem 1.** If $\Gamma$ is well ordered, then $\mathfrak{B}$ can be ordered.

**Proof.** $G$ is isomorphic to the restricted direct sum of the $d$-closed subgroups $D_{\gamma}$ of the reals, where $D_{\gamma} \cong G_{\gamma}/G_{\gamma}$ [1, pp. 19 and 8]. Thus by the corollary to Lemma 2, $\mathfrak{A}$ is isomorphic to the group of matrices $T(\Gamma, D_{\gamma})$. Clearly $[\pi_{\alpha \beta}] \in T$ corresponds to an element of $\mathfrak{B}$ if and only if $\pi_{\gamma \gamma} = 1$ for all $\gamma \in \Gamma$. $0 = G_1 \subset G_2 \subset G_3 = \cdots \subset G_{\omega} \subset G_{\omega} = \cdots$. Thus for each $\gamma \in \Gamma$, $\mathfrak{B}_{\gamma}$ is isomorphic to the group of all matrices of the form

$$
\begin{bmatrix}
1 & & & \\
& a_{21} & 1 & \theta \\
& & & \\
& & & \\
& & a_{\gamma 1} & \cdots & 1 \\
& & & & \theta \\
& & & & \\
& & & & 1
\end{bmatrix}
$$

Therefore without loss of generality $1 = \mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \mathfrak{B}_3 = \cdots \subset \mathfrak{B}_{\omega} \subset \mathfrak{B}_{\omega} = \cdots$. If $\gamma \in \Gamma$ has no immediate predecessor, then $\mathfrak{B}_{\gamma} = \bigcup_{\alpha < \gamma} \mathfrak{B}_{\alpha}$. Also $\mathfrak{B} = \bigcup_{\gamma \in \Gamma} \mathfrak{B}_{\gamma}$. It follows at once from the corollary to Lemma 4 that $\mathfrak{B}$ can be ordered.
Note that the theorem remains true without the restriction that $G$ is $d$-closed. For if $G$ has well ordered rank (that is, $\Gamma$ is well ordered), then so does the $d$-closure of $G$.

**Corollary I.** If $A$ is an abelian $o$-group with well ordered rank, and $\mathfrak{A}^*$ is the group of all $o$-automorphisms of $A$ that induce the identity automorphism on each of the components of $A$, then $\mathfrak{A}^*$ can be ordered.

Let $\pi$ be a value preserving $o$-isomorphism of $G$ into $G$. That is, $g \in G^\gamma \setminus G_\gamma$ if and only if $g \pi \in G^\gamma \setminus G_\gamma$. Define $(G_\gamma + g) \pi' = G_\gamma + g \pi$ for all $g \in G_\gamma$. Then $\pi'$ is an $o$-isomorphism of $G^\gamma / G_\gamma$ into $G^\gamma / G_\gamma$. $T(\Gamma, D_\gamma)$ is isomorphic to the semigroup of all value-preserving $o$-isomorphisms of $G$ into $G$ that induce an $o$-automorphism on each of the components $G^\gamma / G_\gamma$ of $G$. In the proof of the theorem it was shown that $T = U$.

**Corollary II.** If $\Gamma$ is well ordered, then every value preserving $o$-isomorphism of $G$ into $G$ that induces an $o$-automorphism on each of the components of $G$ is an $o$-automorphism of $G$.

**Remark.** Corresponding to Lemma 2 we have the following: Every value preserving $o$-isomorphism of $G$ into $G$ is an $o$-automorphism if and only if $\Gamma$ is well ordered and no $G^\gamma / G_\gamma$ admits an $o$-isomorphism onto a proper subgroup of itself. It is not necessary to assume that $G$ is $d$-closed, since both conditions imply that $G$ is $d$-closed. The proof is a consequence of Lemma 2, and the embedding theorem [1, p. 19].

If $\Gamma$ is well ordered, then by the corollary to Lemma 2, $\mathfrak{A}/\mathfrak{B}$ is isomorphic to the group of all diagonal matrices in $T(\Gamma, D_\gamma)$. The $o$-automorphism group of the subgroup $D_\gamma$ of the reals is isomorphic to a subgroup of the positive reals, hence it has a natural order. Therefore $\mathfrak{A}/\mathfrak{B}$ can be ordered. Thus an ordering of $\mathfrak{B}$ can be extended to an ordering of $\mathfrak{A}$ so that $\mathfrak{B}$ is a convex subgroup if and only if $\alpha^{-1}\beta\alpha$ is positive for $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$ whenever $\beta$ is positive in $\mathfrak{B}$ (Levi [4]). In §4 we show that $\mathfrak{B}$ can be so ordered, provided that the components of $G$ admit only rational $o$-automorphisms. In §3 we order $\mathfrak{B}$ by the method we used to order $\mathfrak{B}$. 

3. **The group $\mathfrak{A}$.** Throughout this section assume that $G$ is a $d$-closed abelian $o$-group with well ordered rank. Let $\mathfrak{A}^*_\gamma(\Gamma)$ be the group of all $o$-automorphisms of $G^\gamma(\gamma)$. $G^\gamma = G_\gamma \oplus D_\gamma$. Thus $\mathfrak{A}^*_\gamma$ is isomorphic to the multiplicative group $\mathfrak{M}$ of all matrices of the form

$$[a] = \begin{bmatrix} a_{11} & \theta \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{11} \in \mathfrak{A}^*_\gamma$, $a_{21} \in H(D_\gamma, G_\gamma)$ and $a_{22}$ is an $o$-automorphism of $D_\gamma$. 

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In particular, \( \mathcal{A}_\gamma \) is isomorphic to a subgroup of \( \mathcal{A}_\gamma \). Suppose that \( \mathcal{A}_\gamma \) and \( H \) are ordered and define \([a]\) positive if \( a_{22} > 1 \) or \( a_{22} = 1 \) and \( a_{11} > 1 \) or \( a_{22} = 1, a_{11} = 1 \) and \( a_{21} > \theta \). Then it is easy to verify that for any non-zero matrix \( a \) in \( \mathbb{M} \), either \( a \) is positive or \( a^{-1} \) is positive. Also if \( a \) and \( b \) are positive matrices in \( \mathbb{M} \), then \( ab \) is positive. Finally suppose that \( a, b \in \mathbb{M} \) and \( b \) is positive.

\[
a^{-1}ba = \begin{bmatrix}
-1 & 0 & \cdots & 0 \\
0 & a_{11}^{-1}b_{11}a_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{22}^{-1}(b_{21}a_{11} + b_{22}a_{21} - a_{21}a_{11}^{-1}b_{11}a_{11})
\end{bmatrix}.
\]

If \( b_{22} > 1 \) or \( b_{22} = 1 \) and \( b_{11} > 1 \), then \( a^{-1}ba \) is positive. Suppose that \( b_{22} = 1, b_{11} = 1 \) and \( b_{21} > \theta \). Then

\[
a^{-1}ba = \begin{bmatrix}
0 & \theta \\
\theta & 0
\end{bmatrix}.
\]

Thus if we wish to extend the order of \( \mathcal{A}_\gamma \) to an order of \( \mathcal{A}_\gamma \) we must order \( H(D_\gamma, G_\gamma) \) so that \( a^{-1}b_{21}a_{11} \) is positive. Now without loss of generality, \( D_\gamma \) is a \( d \)-closed subgroup of the reals. Hence the only \( o \)-automorphisms of \( D \) are multiplications by some positive real numbers. In particular, a multiplication by a positive rational is an \( o \)-automorphism, since \( D \) is \( d \)-closed.

**Lemma 5.** If the only \( o \)-automorphisms of \( D_\gamma \) are multiplications by positive rationals, then \( H(D_\gamma, G_\gamma) \) can be ordered so that if \( h \in H \) is positive, then \( \alpha h \beta \) is positive for all \( o \)-automorphisms \( \alpha \) and \( \beta \) of \( D_\gamma \) and \( G_\gamma \) respectively.

**Proof.** Choose and well order a basis \( d_1, d_2, \cdots \) for \( D \). Define \( h \in H \) positive if \( h \neq \theta \) and \( d_i h > 0 \), where \( d_i \) is the first basis element for which \( d_i h \neq 0 \). \( d_i \alpha h \beta = (rd_i) h \beta = (r(d_i h)) \beta \), where \( r \) is a positive rational and \( \alpha \) and \( \beta \) are \( o \)-automorphisms of \( D_\gamma \) and \( G_\gamma \). Thus \( d_i \alpha h \beta \) is positive if and only if \( d_i h \) is positive.

**Theorem 2.** If \( G \) is an abelian \( d \)-closed \( o \)-group with well ordered rank and each component \( G_\gamma / G_\gamma \) has its group of \( o \)-automorphisms isomorphic to the positive rationals, then \( \mathcal{A} \) can be ordered so that \( \mathcal{B} \) is a convex subgroup.

The proof is entirely similar to the proof of Theorem 1.

**Corollary.** Any torsion free abelian group can be ordered in such a way that the resulting group of \( o \)-automorphisms can also be ordered.

**Proof.** Let \( A \) be a torsion free abelian group. Choose and well order a basis \( a_1, a_2, \cdots \) for the \( d \)-closure \( A^* \) of \( A \). Then \( A^* \)
Define $a = \cdots + r_i a_i + \cdots$ positive if $a \neq 0$ and the nonzero coefficient $r_i$ with greatest subscript is positive. Then $A^*$ satisfies the hypotheses of the theorem. Hence the group of $\sigma$-automorphisms $\mathfrak{A}^*$ of $A^*$ can be ordered, and the group $\mathfrak{A}$ of $\sigma$-automorphisms of $A$ is isomorphic to a subgroup of $\mathfrak{A}^*$.

**Remark.** Note that $\mathfrak{A}^*$ is isomorphic to the group $K$ of all row finite triangular matrices $[\pi_{a\beta}]$, where $\alpha, \beta \in \Gamma$ the rank of $A^*$, $\pi_{aa}$ is a positive rational, $\pi_{a\beta}$ is a rational, and $\pi_{a\beta} = 0$ for $\alpha < \beta$. Then $K$ can be ordered. In fact, the following definition orders $K$. $[\pi_{a\beta}]$ is positive if (a) there exists an $\alpha \in \Gamma$ such that $\pi_{aa} > 1$ and $\pi_{a\beta} = 1$ for all $\beta < \alpha$, or (b) $\pi_{aa} = 1$ for all $\alpha \in \Gamma$ and there exist $\alpha, \beta \in \Gamma$ such that $\pi_{a\beta} > \theta$, all elements above $\pi_{a\beta}$ in the diagonal containing $\pi_{a\beta}$ and parallel to the main diagonal are zero, and all elements in the diagonals parallel to but distinct from the main diagonal and above the diagonal containing $\pi_{a\beta}$ are zero.

4. A generalization of Theorem 2. Our proof of Theorem 2 does not permit the dropping of the hypothesis that $G$ is $d$-closed. For example, if one of the components of $G$ is isomorphic to $3 \oplus 3^{2^{1/2}}$, where $3$ is the group of integers, then the corresponding components of the $d$-closure of $G$ is isomorphic to $\mathfrak{N} \oplus \mathfrak{N}^{2^{1/2}}$. The $\sigma$-automorphism group of $\mathfrak{N} \oplus \mathfrak{N}^{2^{1/2}}$ is isomorphic to the group of all positive elements of $\mathfrak{N} \oplus \mathfrak{N}^{2^{1/2}}$.

**Theorem 3.** If $G$ is an abelian $\sigma$-group with well ordered rank, and each component $G^\gamma/G^\gamma$ has its group of $\sigma$-automorphisms isomorphic to a subgroup of the positive rationals, then $\mathfrak{A}$ can be ordered.

**Proof.** Well order $\prec$ the elements of $G$ so that if $a, b \in G$ and $V(a) < V(b)$, then $a \prec b$, where $V(a)$ is the value of $a$ with respect to the given order of $G$. That is,

$$0 \prec g_{11} \prec g_{12} \prec \cdots \prec g_{21} \prec g_{22} \prec \cdots \prec g_{\omega 1} \prec g_{\omega 2} \prec \cdots \cdots \cdots$$

For each $\pi \neq 1$ in $\mathfrak{A}$ there exists a least element $g$ in the well ordering $\prec$ such that $g\pi \neq g$. Denote this least element by $L(\pi)$, and define $\pi$ positive if $L(\pi)\pi > L(\pi)$. If $\pi \neq 1$ and $\pi$ is not positive, then $L(\pi^{-1}) = L(\pi)$ and $L(\pi)\pi < L(\pi)$. Thus $L(\pi^{-1}) = L(\pi) < L(\pi)\pi^{-1} = L(\pi^{-1})\pi^{-1}$, hence $\pi^{-1}$ is positive. Let $\alpha$ and $\beta$ be positive elements of $\mathfrak{A}$. If $L(\alpha) = L(\beta)$ or $L(\alpha) \prec L(\beta)$ and $g \prec L(\alpha)$, then $g\alpha\beta = g\beta = g$ and $L(\alpha)\alpha\beta > L(\alpha)\beta \geq L(\alpha)$. If $L(\beta) \prec L(\alpha)$ and $g \prec L(\beta)$, then $g\alpha\beta = g\beta = g$ and $L(\beta)\alpha\beta = L(\beta)\beta > L(\beta)$. Therefore $\alpha\beta$ is positive.

Finally suppose that $\alpha, \beta \in \mathfrak{A}$ and that $\beta$ is positive. Let $g \in G^\gamma \setminus G^\gamma$. 

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Then \( g\alpha \equiv (m/n)g \mod G_\gamma \), hence \( n(g\alpha) = mg + d \) where \( m \) and \( n \) are positive integers and \( d \in G_\gamma \). In particular, \( d \geq g \). If \( g \geq L(\beta) \), then 
\[
 n(g\alpha\beta^{-1}) = (mg + d)\beta^{-1} = (mg + d)\alpha^{-1} = ng, \quad \text{hence } g\alpha\beta^{-1} = g. 
\]
If \( g = L(\beta) \), then 
\[
 n(L(\beta)\alpha\beta^{-1}) = (mL(\beta) + d)\beta^{-1} = (mL(\beta)\beta + d)\alpha^{-1} > (mL(\beta) + d)\alpha^{-1} = nL(\beta), \quad \text{hence } L(\beta)\alpha\beta^{-1} > L(\beta). 
\]
Therefore \( \alpha\beta^{-1} \) is positive and \( \mathcal{A} \) is ordered.

Note that for this ordering of \( \mathcal{A} \), \( \mathcal{B} \) is not a convex subgroup. However, the above definition does order \( \mathcal{B} \) for any abelian \( \sigma \)-group with well ordered rank. Thus we have a second proof of Theorem 1. Moreover, this ordering of \( \mathcal{B} \) can be extended to an ordering of \( \mathfrak{A} \) provided that each component of \( G \) has its group of \( \sigma \)-automorphisms isomorphic to a subgroup of the positive rationals. For if \( \alpha \in \mathfrak{A}, \beta \in \mathfrak{B} \) and \( \beta \) is positive, then the above proof shows that \( \alpha\beta\alpha^{-1} \) is positive. Thus we have a proof of Theorem 2 that does not use the hypothesis that \( G \) is \( d \)-closed.

**Bibliography**


Tulane University