ORDER-COMPATIBLE TOPOLOGIES ON A PARTIALLY ORDERED SET

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1. Introduction. Let $X$ be a partially ordered set (poset) with respect to a relation $\leq$, and possessing least and greatest elements $0$ and $I$ respectively. There are many known ways of using the order properties of $X$ to define an "intrinsic" topology on $X$. It is our purpose in this note, instead of considering certain special topologies of this type, to introduce a class of topologies on $X$ which are compatible, in a natural sense, with its order. To this end, let us call a subset $S$ of $X$ up-directed (down-directed) if and only if for all $x \in S$ and $y \in S$ there exists $z \in S$ with $z \geq x, z \geq y(z \leq x, z \leq y)$. Also, following McShane [3], we shall call a subset $K$ of $X$ Dedekind-closed if and only if whenever $S$ is an up-directed subset of $K$ and $y = \text{l.u.b.}(S)$, or $S$ is a down-directed subset of $K$ and $y = \text{g.l.b.}(S)$, we have $y \in K$. We now introduce the following definition, which seems to be a natural requirement for a topology on $X$ to be harmoniously related to its order structure.

Definition. If $\mathcal{J}$ is a topology defined on $X$, we shall say that $\mathcal{J}$ is order-compatible with $X$ if and only if

(i) every set closed with respect to $\mathcal{J}$ is Dedekind-closed, and

(ii) every set of the form $\{x \in X | a \leq x \leq b\}$ is closed with respect to $\mathcal{J}$.

The main purpose of this note is to obtain a simple sufficient condition for a poset $X$ to possess a unique order-compatible topology. We say that two elements $x$ and $y$ in $X$ are incomparable if and only if $x \not\leq y$ and $x \not\geq y$. Let us call a subset $S$ of $X$ diverse if and only if $x \in S$, $y \in S$, and $x \neq y$ imply that $x$ and $y$ are incomparable. We define the width of $X$ to be the l.u.b. of the set $\{k | k$ is the cardinal number of a diverse subset of $X\}$. We shall then prove, as our main result, that a poset of finite width possesses a unique order-compatible topology, with respect to which it is a Hausdorff topological space.

2. Preliminary definitions and lemmas. The reader may verify that the class of all Dedekind-closed subsets of a poset $X$ is closed with respect to arbitrary intersections and finite unions. Hence we may define a topology $\mathcal{D}$ on $X$ whose closed sets are precisely the Dedekind-closed subsets of $X$. We let $\mathcal{S}$ denote the well-known interval topology on $X$, which is obtained by taking all sets of the form $[a, b]$.
= \{ x | a \leq x \leq b \} as a sub-basis for the closed sets. If \( \mathcal{S} \) and \( \mathcal{T} \) are any topologies on \( X \), we define \( \mathcal{S} \leq \mathcal{T} \) to mean that every \( \mathcal{S} \)-closed set is \( \mathcal{T} \)-closed. It is then obvious that we have

**Lemma 1.** If \( \mathcal{T} \) is any order-compatible topology on \( X \), then \( \mathcal{S} \leq \mathcal{T} \leq \mathcal{D} \).

**Lemma 2.** If \( X \) contains no infinite diverse set, then \( X \) is a Hausdorff space in its interval topology.

**Proof.** Suppose \( a \) and \( b \) are any distinct points of \( X \). Then \([4]\) \( X \) is a Hausdorff space in its interval topology if there is a covering of \( X \) by means of a finite number of closed intervals such that no interval contains both \( a \) and \( b \). We consider the following cases, and produce such a covering in each instance.

Case (i). \( a \) and \( b \) are incomparable. Let \( S \) be a maximal diverse subset of \( X \) containing both \( a \) and \( b \). Consider all intervals of the form \([0, s]\) and \([s, I]\) for \( s \in S \). This is a finite set of intervals satisfying the above requirements.

Case (ii). \( a < b \), but \( a < x < b \) for no \( x \in X \). Let \( S \) be a maximal diverse subset of \( X \) containing \( a \), and let \( T \) be a maximal diverse set containing \( b \). Consider the following collections of intervals:

1. all intervals of the form \([0, s]\) for \( s \in S \),
2. all intervals of the form \([t, I]\) for \( t \in T \),
3. all intervals which may exist of the form \([s, t]\) for \( s \in S \) and \( t \in T \), provided that \( s = a \) and \( t = b \) are not both true.

The union of the above three collections of intervals satisfies our requirements.

Case (iii). \( a < b \) and there exists \( x_0 \) with \( a < x_0 < b \). Let \( S \) be a maximal diverse subset containing \( x_0 \), \( T \) a maximal diverse subset containing \( b \). Then the union of the following three collections of intervals satisfies our requirements:

1. all intervals of the form \([0, s]\) for \( s \in S \),
2. all intervals of the form \([t, I]\) for \( t \in T \),
3. all intervals which may exist of the form \([s, t]\) for \( s \in S \), \( t \in T \).

Since the above three cases dispose of all possibilities, the proof is complete.

We shall find it convenient to consider nets of elements in \( X \). We shall follow the terminology of Bartle [1] and Kelley [2], but give all the relevant definitions. If \( f \) is a function defined on an arbitrary up-directed poset \( A \) and with values lying in \( X \), then we say that \( f \) is a net on \( A \) to \( X \). We shall use the notation \((f(\alpha), \alpha \in A)\) for such a net. A net \((g(\beta), \beta \in B)\) is said to be a subnet of \((f(\alpha), \alpha \in A)\) if and only if there is a mapping \( \pi : B \to A \) which satisfies
(i) \( g(\beta) = f(\pi(\beta)) \) for all \( \beta \in B \), and

(ii) given any \( \alpha_0 \in A \), there exists \( \beta_0 \in B \) such that if \( \beta \geq \beta_0 \) then \( \pi(\beta) \geq \alpha_0 \).

Let us call a subset of \( A \) of the form \( A_\beta = \{ \alpha \in A \mid \alpha \geq \beta \} \) a residual subset of \( A \). A subset \( C \) of \( A \) will be called cofinal in \( A \) if and only if \( \alpha \in A \) implies there exists \( \gamma \in C \) with \( \gamma \geq \alpha \). If \( f \) is a net on \( A \) to \( X \), and \( A_\beta \) is a residual subset of \( A \), then the net \( (f(\alpha), \alpha \in A_\beta) \) will be called a residual subnet of \( f \). If \( C \) is cofinal in \( A \), then the net \( (f(\alpha), \alpha \in C) \) will be called a cofinal subnet of \( f \). If \( \beta \in A \), we shall write \( E_f(\beta) \) (or simply \( E(\beta) \), if no confusion can arise) to denote the set \( \{ x \in X \mid x = f(\alpha) \text{ for some } \alpha \geq \beta \} \). A net \( f \) on \( A \) to \( X \) is said to be universal if and only if given any subset \( S \subseteq X \) then either (i) there exists \( \beta \in A \) such that \( E(\beta) \subseteq S \), or (ii) there exists \( \beta \in A \) such that \( E(\beta) \subseteq S' \), the complement of \( S \) with respect to \( X \). It is a well-known result [1; 2] that every net possesses a subnet which is universal.

Now let \( \mathcal{T} \) be any topology on \( X \). We say that a net \( f \) on \( A \) to \( X \) converges to an element \( y \) in \( X \) if and only if for any \( \mathcal{T} \)-open set \( U \) containing \( y \), there exists \( \beta \in A \) such that \( E(\beta) \subseteq U \). If \( f \) converges to \( y \), we write \( f(\alpha) \to y \). A subset \( S \) of \( X \) is closed with respect to \( \mathcal{T} \) if and only if whenever \( f \) is a net whose range is in \( S \) and \( f(\alpha) \to y \), then \( y \in S \) [2, p. 66].

The following notation will be useful. If \( S \subseteq X \), we write \( S^* = \{ x \in X \mid x \geq s \text{ for all } s \in S \} \), and \( S^+ = \{ x \in X \mid x \leq s \text{ for all } s \in S \} \). If \( f \) is a net on \( A \) to \( X \), let \( P_f \) be the union of all sets of the form \( \{ E(\beta) \}^+ \), for some \( \beta \in A \); and let \( Q_f \) be the union of all sets of the form \( \{ E(\beta) \}^* \), for some \( \beta \in A \). Then we say that an element \( y \) in \( X \) is medial for \( f \) if and only if \( y \in P_f \cap Q_f^+ \). We shall need the following lemma, which was proved by Ward [5, Lemma 1] using the terminology of filters.

**Lemma 3 (Ward).** If \( f \) is a net with range in \( X \), and if \( f \) converges to \( y \) in the interval topology on \( X \), then \( y \) is medial for \( f \).

3. **Main results.** Our main theorem will follow as a consequence of three more lemmas.

**Lemma 4.** Let \( f \) be a net on \( A \) to \( X \) and suppose that \( f(\alpha) \to y \) in the interval topology on \( X \). If \( f(\alpha) \) is incomparable with \( y \) for all \( \alpha \in A \), then there exists an infinite diverse subset of \( X \) contained in the range of \( f \).

**Proof.** Let \( (u(\alpha), \alpha \in D) \) be a universal subnet of \( f \). Since every subnet of a convergent net is convergent, and to the same limit, we
have $u(\alpha) \to y$ in the interval topology on $X$. By Lemma 3, $y$ is medial for $u$.

We shall construct inductively an infinite diverse subset of $X$. Select $\delta_1 \in D$ arbitrarily. Since $y \in P_u^*$ and $u(\delta_1)$ is incomparable with $y$, we must have $u(\delta_1) \notin P_u$. Hence the set $K_1 = \{x \in X \mid x \geq u(\delta_1)\}$ contains no $E_u(\alpha)$ for any $\alpha \in D$. Since $u$ is a universal net, there exists some $\alpha_1 \in D$ such that $\alpha_1 > \delta_1$ and $E_u(\alpha_1) \subset K_1' = \{x \in X \mid x \leq u(\delta_1)\}$. Also, since $y \in Q^+$, we have $u(\delta_1) \in Q_u$, and hence $L_1 = \{x \in X \mid x \leq u(\delta_1)\}$ contains no $E_u(\alpha)$ for any $\alpha \in D$. Hence there exists some $\beta_1 \in D$ such that $\beta_1 > \delta_1$ and $E_u(\beta_1) \subset L_1' = \{x \in X \mid x \leq u(\delta_1)\}$. Select $\gamma_1 \in D$ such that $\gamma_1 \geq \alpha_1, \gamma_1 \geq \beta_1$. Then $E_u(\gamma_1) \subset E_u(\alpha_1) \cap E_u(\beta_1)$. It is clear from our construction that $u(\delta_1)$ is incomparable with each element of $E_u(\gamma_1)$. Now choose $\delta_2 \in D$ such that $\delta_2 \geq \gamma_1$. In an analogous way we obtain $\alpha_2$ and $\beta_2$ such that $E_u(\alpha_2) \subset \{x \in X \mid x \leq u(\delta_2)\}$, $E_u(\beta_2) \subset \{x \in X \mid x \leq u(\delta_2)\}$, and $\alpha_2 > \delta_2, \beta_2 > \delta_2$. Then choose $\gamma_2 \in D$ such that $\gamma_2 \geq \alpha_2, \gamma_2 \geq \beta_2$. Then each element of $E_u(\gamma_2)$ is incomparable with both $u(\delta_1)$ and $u(\delta_2)$. Select $\delta_3 \geq \gamma_2$. Continuing in the above manner we obtain an infinite sequence of distinct elements $u(\delta_1), u(\delta_2), u(\delta_3), \ldots$, which form a diverse subset of $X$.

**Lemma 5.** Let $f$ be a net on $A$ to $X$, let $S$ be the range of $f$, and suppose that $y$ is medial for $f$. If $f(\alpha) < y$ for all $\alpha \in A$, then $y = \limsup(S)$.

**Proof.** Suppose that there exists $z \in S^*$ with $z \not\preceq y$. Since $z \in \{E_f(\alpha)\}^*$ for all $\alpha \in A$, we have $z \in Q_f$. But $y \in Q_f^+$, and hence we have a contradiction.

The obvious dual formulation of the above lemma, and also that of the following one, may be left to the reader.

**Lemma 6.** Let $X$ be a poset of finite width, and let $f$ be a net on $A$ with range $(f) = S \subseteq X$. Let $y$ be an element of $X$ such that $y$ is the l.u.b. of the range of every subnet of $f$. Then there exists an up-directed set $M \subseteq S$ such that $y = \liminf(M)$.

**Proof.** Let $k = \text{width of } X$. Let us suppose that the lemma is false. We shall proceed to obtain a contradiction by constructing a diverse subset of $X$ containing $k + 1$ elements.

It is an easy consequence of Zorn's Lemma that every up-directed subset of a poset is contained in a maximal up-directed subset. Let $M_1$ be any maximal up-directed subset of $S$. By our assumption that the lemma is false, we must have $y \not\preceq \liminf(M_1)$. Hence there exists no subnet of $f$ with range contained in $M_1$. Therefore there exists $\alpha_1 \in A$ such that $E(\alpha_1) \subseteq S - M_1$. Now let us choose a maximal up-directed
subset $M_2$ of $E(\alpha_1)$. Since by assumption there exists no subnet of $(f(\alpha), \alpha \in A_{\alpha})$ with range contained in $M_2$, then there is an $\alpha_2 > \alpha_1$ and $E(\alpha_2) \subset E(\alpha_1) - M_2$. Now choose $M_3$, a maximal up-directed subset of $E(\alpha_2)$, and continue the above process for $k$ steps. We obtain sets $M_1$, $M_2$, \cdots, $M_k$; and $E(\alpha_1)$, $E(\alpha_2)$, \cdots, $E(\alpha_k)$, such that (with the agreement that $E(\alpha_0) = S$) $M_i$ is a maximal up-directed subset of $E(\alpha_{i-1})$ and $E(\alpha_i) \subset E(\alpha_{i-1}) - M_i$, for $i = 1$, 2, \cdots, $k$.

Next let us note that, for each $i = 1, 2, \cdots, k$, $x \in E(\alpha_{i-1}) - M_i$ implies (i) $x \not\leq M_i$, and (ii) $x \not\leq m$ for any $m \in M_i$. For if either (i) or (ii) failed to hold, then the set $M_i \cup \{x\}$ would be an up-directed subset of $E(\alpha_{i-1})$, thus contradicting the maximality of $M_i$. Thus for each $x \in E(\alpha_{i-1}) - M_i$ there exists $x_i \in M_i$ such that $x$ and $x_i$ are incomparable.

Now choose an arbitrary element, which we denote by $x_{k+1}$, of $E(\alpha_k) - M_k$. By the above paragraph, there exists $x_k \in M_k$ such that $x_k$ and $x_k$ are incomparable. Also, since $x_k \in E(\alpha_{k-2}) - M_{k-1}$, there exist $a_1 \in M_{k-1}$ and $a_2 \in M_{k-1}$ such that $a_1$ and $x_k$ are incomparable, $a_2$ and $x_k$ are incomparable. Let $x_{k-1}$ be an element of $M_{k-1}$ with $x_{k-1} \geq a_1$, $x_{k-1} \geq a_2$. Then $x_{k-1}$ is incomparable with both $x_k$ and $x_{k+1}$, so that the set $\{x_{k+1}, x_k, x_{k-1}\}$ is diverse. Continuing in this way, we select elements $b_1, b_2, b_3$ in $M_{k-2}$ such that $b_1$ and $x_{k-1}$, $b_2$ and $x_k$, $b_3$ and $x_{k+1}$ form incomparable pairs. Let $x_{k-2}$ be an element of $M_{k-2}$ with $x_{k-2} \geq b_i$ ($i = 1, 2, 3$). Then $\{x_{k+1}, x_k, x_{k-1}, x_{k-2}\}$ is a diverse set. It is clear that continuing the above construction leads to a diverse set $\{x_{k+1}, x_k, \cdots, x_1\}$ of $k + 1$ distinct elements, contained in range $(f)$.

We now have the following theorem.

**Theorem.** If $X$ is a poset of finite width, then $X$ possesses a unique order-compatible topology. Furthermore, with respect to this topology, $X$ is a Hausdorff space.

**Proof.** In view of Lemmas 1 and 2, we need only to prove that the topologies $\sigma$ and $\mathcal{D}$ are equivalent on $X$. Let $K$ be any Dedekind-closed subset of $X$; we shall show that $K$ is $\sigma$-closed. Let $f$ be a net in $K$ with $f(\alpha) \to y$ in the interval topology. We may assume that $f(\alpha) \neq y$ for all $\alpha$. We shall prove that $y \in K$. By Lemma 4, there exists no subnet $g$ of $f$ such that each element of range $(g)$ is incomparable with $y$. Hence there exists a residual subnet of $f$, which we take to be $f$ itself, whose range consists of elements all of which are comparable with $y$. Then there exists (i) a cofinal subnet $u$ of $f$ such that $y$ is an upper bound of range $(u)$, or (ii) a cofinal subnet $v$ of $f$ such that $y$ is a lower bound of range $(v)$. Suppose that (i) holds
1958] TOPOLOGIES ON A PARTIALLY ORDERED SET 529

(the other case is handled in the obvious dual manner). Since \( u \) converges to \( y \) in the interval topology, \( y \) is medial for \( u \) (Lemma 3). Let \( S = \) range \( (u) \). By Lemma 5, \( y = \) l.u.b.(\( S \)). Since every subnet of \( u \) converges to \( y \) in the interval topology, Lemma 6 now applies; and we conclude that there exists an up-directed set \( M \subseteq S \subseteq K \) such that \( y = \) l.u.b.(\( M \)). Since \( K \) was assumed to be Dedekind-closed, we have \( y \in K \), completing the proof.

It is natural to ask whether, in the above theorem, the hypothesis that \( X \) is of finite width can be replaced by the weaker condition that \( X \) contains no infinite diverse subset. However, we have not been able to settle this question (not even in the special case when \( X \) is assumed to be a lattice).

REFERENCES


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