COMMUTATORS IN DIVISION RINGS

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1. Introduction. A theorem of Wedderburn [4] states: if every element of a finite dimensional associative algebra is a sum of nilpotent elements, then the algebra is nil. In [1], Kaplansky asked whether this theorem could be generalized to rings with minimum condition, and mentioned that an equivalent question is:

*Does there exist a division ring D with \( D = [D, D] \)?* (Here \([D, D]\) denotes the subgroup of the additive group of \( D \) generated by all additive commutators \([x, y] = xy - yx\).) An affirmative answer to the second question is equivalent to a negative answer to the first. In the finite dimensional case, the trace function shows that the second question has a negative answer.

In this paper we give an example of a division ring \( D \) with \( D = [D, D] \). In fact every element of \( D \) is itself a commutator, and \( D_n \) (the \( n \times n \) matrix ring with coefficients in \( D \)) has the same property.

2. Construction of the division ring. Ore has shown that a non-commutative integral domain in which every two nonzero elements have a nonzero common right multiple can be imbedded in a division ring of (right) fractions [2]. He has also shown that the integral domain of differential polynomials in one variable with coefficients in a division ring has the common right multiple property. We show that a certain ring of differential polynomials in an infinite number of variables also has the common right multiple property, and that the division ring of fraction satisfies \( D = [D, D] \) and even the stronger properties mentioned above.

Let \( F \) be a field, \( \{x_i\}_{i \in I} \) and \( \{y_i\}_{i \in I} \) two infinite sets of variables with the same ordered index set \( I \), and \( P = F[\{x_i\}, \{y_i\}] \) the set of formal polynomial expressions

\[
\sum a_{n(i_1)} \cdots a_{n(i_k)} m(j_1) \cdots m(j_l) x_{i_1}^{n(i_1)} \cdots x_{i_k}^{n(i_k)} y_{j_1}^{m(j_1)} \cdots y_{j_l}^{m(j_l)}
\]

where \( i_1 < \cdots < i_k, j_1 < \cdots < j_l \) are in \( I \), \( a \cdots \) are in \( F \), and only a finite number of terms occur in the sum. Define addition of polynomials the usual way, and multiplication by \([x_i, y_j] = 0 = [y_i, y_j] \), \([x_i, a] = 0 = [y_i, a] \) for \( a \in F \), \([x_i, y_i] = 1 \), \([x_i, y_j] = 0 \) for \( i \neq j \).

We show \( P \) has the common right multiple property by a method due to Tamari [3]:

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For \( p \in P \), let \( \deg (p) \) denote the total degree of \( p \) in the \( x_i \) and \( y_i \); then \( \deg (pq) = \deg p + \deg q \). Let \( p, q \) be nonzero polynomials, each of degree \( < l \). The problem of finding a common nonzero right multiple \( t = pr = qs \) of degree \( l \) and such that \( r \) and \( s \) each contain only those variables that occur in \( p \) or in \( q \) is the same as the problem of solving a finite number of linear homogeneous equations in a finite number of unknowns: the coefficients of \( r \) and \( s \) are the unknowns, and each term of \( t = pr = qs \) gives an equation. If \( m \) is the larger of \( \deg p, \deg q \) and \( v \) is the number of variables that occur in \( p \) or in \( q \) then the number of equations is the same as the number of distinct monomials of degree \( \leq l \) in \( v \) variables, i.e. \( C_{v+l,v} \), and the number of terms in each of \( r, s \) is \( \geq C_{v+1-m,v} \). Thus we have \( C_{v+l,v} \) equations in at least \( 2C_{v+1-m,v} \) unknowns. If \( l \geq m/(1 - 2^{-1/v}) \), there are more unknowns than equations and a nonzero solution exists.

Let \( D \) be the division ring of fractions \( p/q = pq^{-1}, \ p, q \in P \). If \( d = pq^{-1} \in D \), each of \( p, q \) contains only a finite number of the variables \( y_i \), and so, for some index \( n \), \( y_n \) does not occur in \( p \) or in \( q \). Then \( [x_n, p] = 0 = [x_n, q] \) and \( [x_n, d] = 0 \), \( [x_n, y_n d] = [x_n, y_n] d = d \). Similarly, if \( d_1, \ldots, d_r \) are a finite number of elements of \( D \), then \( y_n \) does not occur in any of the \( d_j \) for some \( n \), and so \( [x_n, y_n d_j] = d_j, j = 1, \ldots, r \). In particular if \((d_{ij})\) is an \( m \times m \) matrix over \( D \), we can find \( n \) such that \( [x_n, y_n d_{ij}] = d_{ij} \), for all \( i, j \). Let \( c_{ij} = y_n d_{ij} \) and let \( (x_n) \) be the matrix \( x_n I, I \) the identity matrix; then \( (x_n), (c_{ij}) = (d_{ij}) \).

3. Commutators and nilpotent matrices. We owe the first proposition and its proof to Professor Kaplansky, and the rest of the section is an amplification of his remarks in [1].

**Proposition 1.** Let \( R \) be any ring, \( R_n \) the \( n \times n \) matrix ring over \( R \), \((n \geq 2)\). If \( A \in R_n \) and trace of \( A \in [R, R] \), then \( A \) is a sum of nilpotent elements. In particular, if \( A \in [R_n, R_n] \) then \( A \) is a sum of nilpotent elements.

**Proof.** Any matrix is the sum of a diagonal matrix (i.e. one with zeros off the main diagonal) and two nilpotent matrices, so that for our purposes only the diagonal elements matter. The following \( 2 \times 2 \) matrices are nilpotent:

\[
\begin{pmatrix}
ab & a \\
-ba & -ba
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-ba & ba \\
ba & ba
\end{pmatrix},
\]

also \( \begin{pmatrix}
-d & -d \\
d & d
\end{pmatrix} \); thus

\[
\begin{pmatrix}
\begin{bmatrix}
ab
\end{bmatrix} & \ast \\
\ast & 0
\end{pmatrix}
\]

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is a sum of nilpotent matrices.

Now let $A = (a_{ij}) \in R_n$, and $\sum_{i} a_{ii} = c \in [R, R]$, so $a_{11} = c - (a_{22} + \cdots + a_{nn})$. Let $xe_{ij}$ denote the matrix with $x$ in the $i, j$ position and zeros elsewhere. Using the above $2 \times 2$ matrices, we see that $ce_{11}$ and $-a_{ii}e_{11} + a_{ii}e_{ii}, i \geq 2$, are sums of nilpotent matrices; therefore this also holds for $a_{11}e_{11} + \cdots + a_{nn}e_{nn}$, and finally for $A$. If $A \in [R_n, R_n]$, then trace of $A \in [R, R]$.

**Proposition 2.** Let $D$ be any division ring, $A$ an element of $D_n$, $n \geq 2$. The following conditions are equivalent:

(a) $A \in [D_n, D_n]$,

(b) trace of $A \in [D, D]$,

(c) $A$ is a sum of nilpotent elements.

**Proof.** We have already shown $a \rightarrow b$, $b \rightarrow c$.

Define a “trace modulo commutators,” $\text{tr} (A)$, as the coset of the ordinary trace of $A$ in the factor group of the additive group of $D$ modulo $[D, D]$. Then $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$, and $\text{tr}(AB) = \text{tr} (BA)$. Now let $A$ be nilpotent: then there exists a nonsingular $B$ such that $BAB^{-1}$ is in Jordan canonical form, i.e. $BAB^{-1}$ is a sum of matrices of the form $n_{k,l} = e_{k+1,k} + e_{k+2,k+1} + \cdots + e_{l+1,l}, 1 \leq k \leq l \leq n - 1$. Finally, $[ \sum_{i} (i+1-k)e_{i,i}, n_{k,l}] = n_{k,l}$; thus $n_{k,l}$ and also $A$ are in $[R_n, R_n]$. This shows (c) $\rightarrow$ (a).

**Corollary.** If $D$ is a division ring such that $D = [D, D]$, then every element of $D_n$, $n \geq 2$, is a sum of nilpotent elements.

**Bibliography**


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