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**SIMPLE NODAL NONCOMMUTATIVE JORDAN ALGEBRAS**

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1. **Introduction.** Nodal algebras were defined by R. D. Schafer [4] and have also been studied by the author [2; 3]. A noncommutative Jordan algebra is an algebra $\mathfrak{A}$ over a field $\mathbb{F}$ satisfying (1) the flexible law $(xy)x=x(yx)$ and (2) the condition that $\mathfrak{A}^+$ is a Jordan algebra. That is, $\mathfrak{A}^+$ satisfies the identity $(x^2y)\cdot x=x^2\cdot (y\cdot x)$ where we have used the dot to indicate the product of $\mathfrak{A}^+$. The algebra $\mathfrak{A}^+$ is defined to be the same vector space as $\mathfrak{A}$ but with product $x\cdot y=(xy+yx)/2$ where $xy$ and $yx$ are products in $\mathfrak{A}$. Then $\mathfrak{A}$ is called nodal if it is finite dimensional, if $\mathfrak{A}$ has identity element 1, if $\mathfrak{A}$ can be written as a vector space direct sum $\mathfrak{A}=\mathfrak{N}+\mathfrak{I}$ where $\mathfrak{N}$ is the subspace of nilpotent elements of $\mathfrak{A}$, and if $\mathfrak{I}$ is not a subalgebra of $\mathfrak{A}$.

Every known nodal algebra $\mathfrak{A}$ has the property that $\mathfrak{A}^+$ is an associative algebra. The flexible algebras with $\mathfrak{A}^+$ associative have been described in [3]. In this paper we shall prove the following theorem.

**Theorem 1.** Let $\mathfrak{A}$ be a simple nodal noncommutative Jordan algebra of characteristic $\neq 2$. Then $\mathfrak{A}^+$ is associative.

Define $\mathfrak{B}$ to be the subspace of $\mathfrak{A}$ generated by all the associators in $\mathfrak{A}^+$. That is, $\mathfrak{B}$ is generated by elements of the form $(x\cdot y)\cdot z-x\cdot (y\cdot z)$ with $x$, $y$, $z$ in $\mathfrak{A}$. The proof of the theorem will be made by showing that the ideal $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B}$ is not all of $\mathfrak{A}$ and since $\mathfrak{A}$ is simple it will follow that $\mathfrak{C}=0$ and $\mathfrak{B}=0$. This is the desired result.

The original proof was not valid when the characteristic is 3. The author thanks Professor R. D. Schafer for suggesting a modification.

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which makes the proof simpler and also valid when the characteristic is 3.

2. The proof. Let \( x_1, x_2, x_3, \) and \( y \) be any elements of \( \mathcal{A} \). Since \( \mathcal{A} \) is nodal we have:

\[
x_iy = \lambda_i x_i + z_i.
\]

The proof will depend on relation (8) of Schafer’s paper [4] which is

\[
(x_1 \cdot x_2)y = \lambda_1 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_1
\]

\[
\quad - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_1 \cdot x_2) \cdot y.
\]

In the proof it will also be necessary to use the fact that \( \mathcal{A}^+ \) is a subalgebra of \( \mathcal{A}^+ [1] \).

By (2) \( (x_1 \cdot x_2) y \) is in \( \mathcal{A} \) and it follows from (2) that \([ (x_1 \cdot x_2) \cdot x_3 ] y = \lambda_3 x_1 \cdot x_2 + (x_1 \cdot x_2) \cdot z_3 + x_3 \cdot [ \lambda_3 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_1 - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_1 \cdot x_2) \cdot y ] - [(x_1 \cdot x_2) \cdot y] \cdot x_3 - (x_3 \cdot y) \cdot (x_1 \cdot x_2) + [(x_1 \cdot x_2) \cdot x_3] \cdot y.
\]

Without bothering to simplify interchange subscripts 1 and 3 to get

\[
[x_1 \cdot (x_2 \cdot x_3)] y = \lambda_1 x_3 \cdot x_2 + (x_3 \cdot x_2) \cdot z_1 + x_1 \cdot [ \lambda_3 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_3 - (x_3 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_2 \cdot x_3) \cdot y ] - [(x_2 \cdot x_3) \cdot y] \cdot x_1 - (x_1 \cdot y) \cdot (x_3 \cdot x_2) + [(x_3 \cdot x_2) \cdot x_1] \cdot y.
\]

Using the notation \( (a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c) \) for the associator of \( a, b, c \) we have, upon subtracting the second relation from the first, \( (x_1, x_2, x_3) y = (x_1, x_2, z_3) + (x_1, z_2, x_3) + (z_1, x_2, x_3) - (x_1 \cdot y, x_2, x_3) - (x_1, x_2 \cdot y, x_3) + (x_3 \cdot y, x_2, x_1) + (x_1, x_2, x_3) \cdot y.
\]

Now define the set \( \mathcal{B} \) to be the subspace of \( \mathcal{A} \) generated by the associators \( (a, b, c) \) with \( a, b, c \) in \( \mathcal{A} \) and using the product of \( \mathcal{A}^+ \). We have proved the following lemma.

**Lemma 1.** Let \( \mathcal{A} \) be a nodal noncommutative Jordan algebra whose characteristic is not 2. Then \( \mathcal{A} \mathcal{N} \subseteq \mathcal{B} + \mathcal{B} \cdot \mathcal{N} \). Also \( \mathcal{N} \mathcal{B} \subseteq \mathcal{B} + \mathcal{B} \cdot \mathcal{N} \).

The last statement follows from the fact that if \( b \) is in \( \mathcal{B} \), \( n \) in \( \mathcal{N} \), then \( nb = 2b \cdot n - bn \).

Let \( \mathcal{C}_0 = \mathcal{B} \), \( \mathcal{C}_1 = \mathcal{B} + \mathcal{B} \cdot \mathcal{N} = \mathcal{C}_0 + \mathcal{C}_0 \cdot \mathcal{N} \), and in general \( \mathcal{C}_{i+1} = \mathcal{C}_i + \mathcal{C}_i \cdot \mathcal{N} \). Equivalently, \( \mathcal{C}_{i+1} = \mathcal{C}_i + \mathcal{B} (R_{\mathcal{A}}^+)^i + 1 \).

**Lemma 2.** The product \( (\mathcal{B} \cdot \mathcal{N}) \mathcal{N} \subseteq \mathcal{C}_2 \) and \( \mathcal{N} (\mathcal{B} \cdot \mathcal{N}) \subseteq \mathcal{C}_2 \). It follows that \( \mathcal{C}_1 \mathcal{N} \subseteq \mathcal{C}_2, \mathcal{N} \mathcal{C}_1 \subseteq \mathcal{C}_2 \).

The proof follows from the flexible law as does (2) which was proved by Schafer. The linearized form of the flexible identity is

\[
(xy)z + (zy)x = x(yz) + z(yx).
\]

Add \( (yx)z + (yz)x \) to both sides of (3) to obtain the equivalent relation

\[
(x \cdot y)z + (y \cdot z)x = yz \cdot x + yx \cdot z.
\]
If $x$ is in $\mathfrak{N}$, $y$, $z$ in $\mathfrak{G}$, then $(y \cdot z)x$ is in $(\mathfrak{N} \cdot \mathfrak{G}) \mathfrak{G} \subseteq \mathfrak{N} \mathfrak{G}$. By Lemma 1, $(y \cdot z)x$ is in $C_1$. The product $yz$ is in $\mathfrak{N} \mathfrak{G} + \mathfrak{N}$ so $yz \cdot x$ is in $\mathfrak{B} + \mathfrak{N} \cdot \mathfrak{B} = C_1$. And $yx \cdot z$ is in $\mathfrak{N} \mathfrak{G} \cdot \mathfrak{N} \subseteq C_2$. Therefore, $(x \cdot y)z$ is in $C_2$ as desired.

**Lemma 3.** The product $[\mathfrak{B}(R^+_I)^i] \mathfrak{N} \subseteq C_{i+1}$ and $\mathfrak{N}[\mathfrak{B}(R^+_I)^i] \subseteq C_{i+1}$. Or, equivalently, $C_i \mathfrak{N} \subseteq C_{i+1}$, $\mathfrak{N} C_i \subseteq C_{i+1}$.

Assume that $[\mathfrak{B}(R^+_I)^i-1] \mathfrak{N}$ and $\mathfrak{N} [\mathfrak{B}(R^+_I)^i-1]$ are in $C_i$. Take $x$ in (4) to be in $S = \mathfrak{B}(R^+_I)^{i-1}$, and $y$, $z$ to be in $\mathfrak{N}$. Then $(y \cdot z)x$ is in $\mathfrak{N} \mathfrak{S} \subseteq C_i$, $yz \cdot x$ is in $\mathfrak{S} + \mathfrak{N} \cdot \mathfrak{S} \subseteq C_i$, and $yx \cdot z$ is in $(\mathfrak{N} \mathfrak{S}) \cdot \mathfrak{N} \subseteq C_{i+1}$. Thus $(x \cdot y)z$ is in $C_{i+1}$.

**Lemma 4.** There exists a positive integer $k$ such that $C_k = C_{k+1}$ and $C_k$ is an ideal of $\mathfrak{A}$.

The set $\mathfrak{B}$ is contained in $\mathfrak{N}$. Since $\mathfrak{N}^+$ is a Jordan algebra, $\mathfrak{N}^+$ is nilpotent. Consequently, $\mathfrak{B}(R^+_I)^{k+1} = 0$ for some $k$. For this $k$, $C_k = C_{k+1}$. By Lemma 3 $C_k \mathfrak{N} \subseteq C_{k+1} = C_k$ and $\mathfrak{N} C_k \subseteq C_k$.

The ideal $C_k$ is contained in $\mathfrak{N} \cdot \mathfrak{N} \cdot \mathfrak{N} = \mathfrak{N}_3$. Since $\mathfrak{N}^+$ is a subalgebra of $\mathfrak{N}^+$, $\mathfrak{N}_3 \subseteq \mathfrak{N}$ and so $C_k \subseteq \mathfrak{N}$. If $\mathfrak{A}$ is a simple algebra, $C_k = 0$ and thus $\mathfrak{B} = 0$. This says that every associator in $\mathfrak{N}^+$ is zero. Now if $a$, $b$, $c$ are any elements in $\mathfrak{A}$, $a = \alpha_1 + x$, $b = \beta_1 + y$, $c = \gamma_1 + z$ with $x$, $y$, $z$ in $\mathfrak{N}$. Then $(a \cdot b) \cdot c - a \cdot (b \cdot c) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ so every associator in $\mathfrak{N}^+$ is an associator in $\mathfrak{N}^+$. This completes the proof of the theorem.

Any ideal properly contained in $\mathfrak{A}$ is contained in $\mathfrak{N}$, hence is a nilideal and is contained in the radical of $\mathfrak{A}$. This implies the corollary which we state below.

**Corollary.** Let $\mathfrak{A}$ be a semisimple nodal noncommutative Jordan algebra of characteristic $\neq 2$. Then $\mathfrak{A}$ is simple and $\mathfrak{N}^+$ is associative.

**References**


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