ON THE EXISTENCE OF WEAK CUT POINTS IN PLANE CONTINUA

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Suppose that K is the sum of two circles in the plane tangent at the point x. (They may be "internally" or "externally" tangent.) Let D be the complementary domain of K having K as its boundary; let Q be one of the other complementary domains; and let M be the complement of $D+Q$. If p is a point of M not in $\overline{Q}$ and q is a point of Q, it is clear that x cuts p from q in $M+Q$. In fact, since M is locally connected, the existence of such a cut point follows from a well-known separating point theorem in plane topology.[2] However, if, instead of being a locally connected continuum, K is merely a continuum (possibly indecomposable), then M is not necessarily locally connected and the existence of such a cut point is not so evident or its ambiguous location gives no clue to the proof of its existence. The purpose of this paper is to state and prove a cut point existence theorem of this general nature. As is frequently the case with separating point theorems, when one weakens the hypothesis by discarding the local connectedness requirement, one must weaken the conclusion by replacing the notion of "separating point" by the notion of "cut point." But since the two notions are equivalent in the presence of local connectedness, one usually gets a stronger theorem, and this is the case here.

DEFINITIONS AND NOTATION. Let *space* be a 2-sphere and let S denote the set of all points of the space.[3] If a and b are points of a continuum M, the subset X of $M-(a+b)$ is said to *cut a from b in M* provided that every subcontinuum of M containing a and b contains a point of X. When X contains only one point, it is called a *cut point*. If M is a continuum, by a *complementary domain of M* is meant a component of $S-M$. If D is a connected open set and Q is a connected subset of $S-\overline{D}$, by the *outer boundary of D with respect to Q* is meant the boundary of the complementary domain of $\overline{D}$ containing Q.

[2] E.g., Theorem 43, p. 222, of [1].
[3] For compact continua, the theorem and proof are applicable to other spaces such as the plane, any space satisfying Axioms 0–5 of [1], etc.

THEOREM.[4] *Suppose that D and Q are different complementary domains of a continuum M such that the boundary B of Q is the outer boundary of D with respect to Q. Then if $p \in M - B$ and $q \in Q$, there exists a point x of B which cuts p from q in $M + Q$.*

PROOF.[5] Let ρ denote a distance function for S. If there exists a simple sequence α of closed subsets of $M + Q$ each of which cuts p from q in $M + Q$ such that α converges to a point x of B, then x cuts p from q in $M + Q$. For otherwise if some subcontinuum H of $M + Q$ contains $p + q$ but not x, there exists a term A of α such that $\rho(A, H)$ is positive and $A \cdot H \neq 0$. This is a contradiction. So if the theorem is false no such sequence α exists.

Now assume that the theorem *is* false. Hence there exists a positive number ϵ such that if L is a closed subset of $M + Q$ whose diameter is 3ϵ or less and whose distance from B is less than ϵ then L does not cut p from q in $M + Q$. This follows from the above paragraph and the compactness of $M + Q$. For convenience it is assumed that ϵ is less than $\rho(p + q, B)$ and also less than the diameter of B.

There exists a simple closed curve J lying in D such that *one* of its two complementary domains (call this domain E) contains M and for each point z of B, $\rho(z, J) < \epsilon/2$. There exists a finite collection G of circular regions covering B such that the center of each element of G belongs to B and each element of G has *diameter* ϵ. Let C_1, $C_2, \cdots, C_n$ denote the boundaries of the elements of G. Since B is a continuum, some subset of $C_1 + C_2 + \cdots + C_n$ is a simple closed curve in E separating B from q; hence $(C_1 + C_2 + \cdots + C_n) \cdot (E + J)$ separates p from q in $E + J$. So there exists a finite collection $T_1, T_2, \cdots, T_m$ of arcs lying in $E + J$ such that (1) for each i, T_i intersects J in only its end points, (2) for each i, there exists a j such that $T_i \subset C_j$ and (3) $\sum T_i$ separates p from q in $E + J$.

Now T_1 does not separate p from q in $E + J$ for if it did $T_1 \cdot (M + Q)$ would cut p from q in $M + Q$, would be of diameter 3ϵ or less, and would lie at a distance of less than ϵ from B contrary to the definition of ϵ. There exists a simple domain E_1 whose boundary J_1 is a subset of $J + T_1$ such that $E_1 \subset E - E \cdot T_1$ and $p + q \subset E_1$. Proceeding by induction, suppose that E_k is a simple domain whose boundary J_k is a subset of $J + T_1 + \cdots + T_k$ such that $E_k \subset E_{k-1} - E_{k-1} \cdot \sum_{i=1}^{k} T_i$ and $p + q \subset E_k$ but such that $T_{k+1} \cdot \overline{E}_k$ does separate p from q in $\overline{E}_k$. (This

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[4] One aspect of how close this theorem comes to being false is illustrated in Fig. 24, p. 223, of [1].

[5] The reader will find Theorems 11–15, pp. 163–167 and Theorems 28–34, pp. 197–203, of [1] helpful in verifying some of the steps in this argument.

must happen for some k less than m, since $\sum T_i$ separates p from q in $E+J$.) It follows[6] that some component T'_{k+1} of $T_{k+1} \cdot \overline{E}_k$ separates p from q in $\overline{E}_k$. This set T'_{k+1} is an arc with its end points a and b in J_k. Now a either belongs to J or, for some natural number $i \leqq k$, belongs to T_i (which has its end points in J); likewise b either belongs to J or, for some natural number $j \leqq k$, belongs to T_j (which has its end points in J). Let pq be an arc in E_k from p to q. Since pq intersects T'_{k+1}, let p' and q' be the first and last points (respectively) of $pq \cdot T'_{k+1}$ in the order from p to q. The subarcs pp' and qq' of pq abut on T'_{k+1} from different sides. In $T_i + T_j + J$ there is an arc T' such that $T' + T'_{k+1}$ is a simple closed curve containing only $p' + q'$ of $pp' + qq'$. Hence $T' + T'_{k+1}$ separates p from q. But $(M+Q) \cdot (T' + T'_{k+1})$ separates p from q in $M+Q$ and is a subset of $T_i + T_j + T_{k+1}$ which is of diameter less than or equal to 3ϵ. Furthermore $\rho(T_i + T_j + T_{k+1}, B) < \epsilon$. This involves a contradiction from which the validity of the theorem follows.

The preceding theorem has been formulated so as to parallel as nearly as possible the corresponding theorem for locally connected continua in [1]. A simpler but equivalent statement (suggested by the referee) follows:

THEOREM. *If the continuum M does not separate S, Q is a component of the interior of M, and p belongs to $M - \overline{Q}$, then there is a point x of $\overline{Q}$ which cuts p from Q in M.*

COROLLARY.[7] *If the continuum M does not separate S, and p and q belong to different components of the interior of M, then some point x of M cuts p from q in M.*

BIBLIOGRAPHY

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2. R. E. Bayse, *Simply connected sets*, Trans. Amer. Math. Soc. vol. 38 (1935) pp. 341–356.

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[6] For instance, see Theorem 13 of [2]. There are other results in this paper of relevant interest.

[7] Proposed by Professor A. M. Gleason (in conversation).