A NOTE ON THE RELATIONSHIP BETWEEN CERTAIN SUBGROUPS OF A FINITE GROUP

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A well-known result of G. Frobenius (cf. [2]) states that if $\mathcal{C}$ is a normal subgroup of the finite group $G$, then an irreducible $G$-module (relative to any base field $\mathfrak{F}$) either remains irreducible as an $\mathcal{C}$-module or decomposes into a direct sum of conjugate irreducible $\mathcal{C}$-modules. Simple examples readily demonstrate that the conclusion of this theorem may hold even though $\mathcal{C}$ is not normal. In §1 a version of the Frobenius result is stated and the converse considered. This opens the question: What is the relationship between a group $G$ and one of its subgroups $\mathcal{C}$ if each irreducible $G$-module over a field $\mathfrak{F}$ remains irreducible as an $\mathcal{C}$-module? It is shown in §2 that for "most" fields $\mathfrak{F}$ (the modular fields naturally cause a certain amount of difficulty) the answer is that $G$ is an extension of $\mathcal{C}$ by an abelian group such that each conjugate class of $\mathcal{C}$ is also a conjugate class of $G$. To determine whether this last property leads to the conclusion that $G$ is the trivial extension of $\mathcal{C}$, extensions are considered in §3 and it is shown that the answer is in general negative. However, using a result due to M. Hall [4] it is proved that this latter property does imply that $G$ is the trivial extension of $\mathcal{C}$ in many cases.

Since results contingent on absolute irreducibility are used in certain proofs, it will be assumed throughout this note that $\mathfrak{F}$ is always a splitting field for every irreducible representation of the groups being discussed.

1. Preliminary remarks. Let $\mathcal{C}$ be a subgroup of the finite group $G$ and let $M$ be a left (right) $G$-module with base field $\mathfrak{F}$. If $N$ is a left (right) $\mathcal{C}$-submodule of $M$ and if $G \subseteq \mathcal{C}$, then submodule $G \cdot N(\cdot N)$ of $M$ is said to be a conjugate of $N$ relative to $G$. Obviously it need not be an $\mathcal{C}$-module.

Now the key to the Frobenius Theorem is the result [2]:

If $\mathcal{C}$ is a normal subgroup of $G$ then an irreducible $G$-module $M$ contains an irreducible $\mathcal{C}$-submodule $N$ which has the property that each conjugate of $N$ relative to $G$ is also an $\mathcal{C}$-module.

Consideration of the converse proposition leads to the following:

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1 It was pointed out by the referee that Theorem 3, for example, may be false if $\mathfrak{F}$ is not a splitting field for every irreducible representation of $G$ and $H$. The symmetric group on three elements, its normal subgroup, and the rational field illustrate this possibility.
Theorem 1. If \( \mathcal{K} \) is a subgroup of \( \mathfrak{G} \) such that each irreducible \( \mathfrak{G} \)-module \( \mathcal{M} \) over a field \( \mathfrak{F} \) contains an irreducible \( \mathfrak{K} \)-module \( \mathcal{U} \) all of whose conjugates relative to \( \mathfrak{G} \) are also \( \mathcal{K} \)-modules, then each irreducible \( \mathcal{K} \)-module remains irreducible as an \( \mathcal{K} \)-module, where \( \mathcal{K} \) is the minimal normal subgroup of \( \mathfrak{G} \) which contains \( \mathcal{K} \).

Let \( \mathcal{M} \) be an irreducible left \( \mathfrak{G} \)-module. Since \( \mathcal{K} \) is normal in \( \mathfrak{G} \), \( \mathcal{M} \) is a direct sum of conjugate irreducible left \( \mathcal{K} \)-modules, \( \mathcal{R}_i \), each of dimension \( m \) relative to \( \mathfrak{F} \): \( \mathcal{M} = \mathcal{R}_1 + \cdots + \mathcal{R}_n \), \( n \geq 1 \). On the other hand, from the hypothesis \( \mathcal{M} \) contains an irreducible left \( \mathcal{K} \)-submodule \( \mathcal{U} \) all of whose conjugates relative to \( \mathfrak{G} \) are also \( \mathcal{K} \)-modules, necessarily irreducible. Now let \( G \subseteq \mathfrak{G}, H \subseteq \mathcal{K} \); then \( GU \) is an \( \mathcal{K} \)-module and therefore \( (G^{-1}HG)\mathcal{U} = G^{-1}H(GU) = G^{-1}(GU) = \mathcal{U} \). So \( \mathcal{U} \), of dimension \( u \) over \( \mathfrak{F} \), is an irreducible \( \mathcal{K} \)-module. Therefore \( m = u \) and since each \( \mathcal{R}_i \) is also a left \( \mathcal{K} \)-module it must remain irreducible as an \( \mathcal{K} \)-module. As every irreducible \( \mathcal{K} \)-module is \( \mathcal{K} \)-isomorphic with a submodule of a \( \mathfrak{G} \)-module, the result follows.

This interesting relationship between \( \mathcal{K} \) and \( \mathfrak{G} \) will be investigated in the remainder of the paper.

2. Property \( \mathfrak{S} \). To simplify matters we introduce the following definition. A subgroup \( \mathcal{K} \) of the group \( \mathfrak{G} \) is said to possess property \( \mathfrak{S} \) relative to the field \( \mathfrak{F} \) if each irreducible \( \mathfrak{G} \)-module over \( \mathfrak{G} \) remains irreducible as an \( \mathcal{K} \)-module.

Theorem 2. If \( \mathcal{K} \) possesses property \( \mathfrak{S} \) relative to \( \mathfrak{G} \), then \( \mathcal{K} \) is normal in \( \mathfrak{G} \) and \( \mathfrak{G}/\mathcal{K} \) is an abelian group if either of the following conditions is satisfied:

(i) The radical \( \mathfrak{R}(\mathfrak{G}) \) of the group algebra \( \mathfrak{A}(\mathfrak{G}) \) of \( \mathfrak{G} \) over \( \mathfrak{F} \) equals \( \mathfrak{A}(\mathfrak{G}) \cdot \mathfrak{K}(\mathcal{K}) \), where \( \mathfrak{K}(\mathcal{K}) \) is the radical of \( \mathfrak{A}(\mathcal{K}) \), the group algebra of \( \mathcal{K} \) over \( \mathfrak{F} \).

(ii) The characteristic of \( \mathfrak{G} \) is \( p \) and \( \mathcal{K} \) is a Sylow \( p \)-subgroup of \( \mathfrak{G} \).

Let \( \mathfrak{F} \) be the ideal of \( \mathfrak{A}(\mathcal{K}) \) which has as its basis the differences \( H_i - H_j \), all \( H_i, H_j \subseteq \mathcal{K} \). Then \( \mathcal{K} \) is normal in \( \mathfrak{G} \) if and only if the left ideal \( \mathfrak{L} = \mathfrak{A}(\mathfrak{G}) \cdot \mathcal{K} \) is a two-sided ideal in \( \mathfrak{A}(\mathfrak{G}) \). Now (i) implies that \( \mathfrak{L} \supseteq \mathfrak{R}(\mathfrak{G}) \), since \( \mathfrak{L} \supseteq \mathfrak{R}(\mathcal{K}) \), so it will be sufficient to show that \( \mathfrak{A}(\mathfrak{G}) = \mathfrak{A}(\mathfrak{G}) - \mathfrak{R}(\mathfrak{G}) \) is an ideal. \( \mathfrak{A}(\mathfrak{G}) \) contains an algebra \( \mathfrak{D} \supseteq \mathfrak{A}(\mathcal{K}) \) and \( \mathfrak{D} = \mathfrak{U} \oplus \mathfrak{B} \) with \( \mathfrak{U} \supseteq \mathfrak{A}(\mathcal{K}) - \mathfrak{G} \) and of dimension one over \( \mathfrak{F} \). Then \( \mathfrak{A}(\mathfrak{G}) = \mathfrak{A}(\mathfrak{G}) \mathfrak{D} = \mathfrak{A}(\mathfrak{G}) \mathfrak{U} + \mathfrak{A}(\mathfrak{G}) \mathfrak{B} \), a direct sum of left ideals of \( \mathfrak{A}(\mathfrak{G}) \), with \( \mathfrak{A}(\mathfrak{G}) \mathfrak{B} \supseteq \mathfrak{G} \). But \( \mathfrak{A}(\mathfrak{G}) \mathfrak{U} \) and \( \mathfrak{G} \) are right \( \mathcal{K} \)-modules, so if \( \mathfrak{B} \) is a minimal right ideal of \( \mathfrak{A}(\mathfrak{G}) \), hence an irreducible right \( \mathcal{K} \)-module, it must lie entirely in \( \mathfrak{A}(\mathfrak{G}) \mathfrak{U} \) or \( \mathfrak{G} \). Hence \( \mathfrak{G} \)}
is also a right ideal of $\mathfrak{A}(G)$ and so $\mathfrak{K}$ is normal in $G$. Furthermore $G/\mathfrak{K}$ is represented isomorphically over $\mathfrak{A}(G) = \mathfrak{A}(G)/\mathfrak{A}(\mathfrak{K}) \cup$ which is necessarily a sum of fields since $\cup$ is one dimensional.

If (ii) is satisfied then all the irreducible representations of $G$ are one dimensional since the only irreducible representation of $\mathfrak{K}$ is the identity representation. Therefore there exists a minimal normal subgroup $\mathfrak{K}$ such that $G/\mathfrak{K}$ is abelian and $\mathfrak{A}(G/\mathfrak{K})$ is semisimple. It follows simply (cf. [3]) that $\mathfrak{K}$ is necessarily of order $p^a$ and hence $\mathfrak{K} = \mathfrak{K}$.

If $G$ is restricted so that $\mathfrak{A}(G)$ is semisimple then the following deeper result may be obtained.

**Theorem 3.** If $\mathfrak{K}$ is a subgroup of $G$ possessing property $\mathfrak{A}$ relative to the field $\mathfrak{F}$ of characteristic $0$ or $p$, $(p, o(G)) = 1$, then each conjugate class of $\mathfrak{K}$ is also a conjugate class in $G$.

Let $\mathfrak{C}(G)$ and $\mathfrak{C}(\mathfrak{K})$ be the centers of $\mathfrak{A}(G)$ and $\mathfrak{A}(\mathfrak{K})$ respectively. We must show that $\mathfrak{C}(\mathfrak{K})$ is a subalgebra of $\mathfrak{C}(G)$. Let $\mathfrak{L}$ be a minimal left ideal of $\mathfrak{A}(G)$; hence it is an irreducible left $\mathfrak{K}$-module and so there exists a primitive idempotent $e \in \mathfrak{C}(\mathfrak{K})$ such that $e\mathfrak{A}(\mathfrak{K})\mathfrak{L} = \mathfrak{L}$. Now $\mathfrak{A}(G) = \mathfrak{A}(\mathfrak{K}) \cdot \mathfrak{A}(G)$ and $e\mathfrak{A}(\mathfrak{K})\mathfrak{A}(G) \supset \mathfrak{L}$, so if $\mathfrak{L}$ is the set of all minimal left ideals $\mathfrak{L}$ of $\mathfrak{A}(G)$ such that $e\mathfrak{A}(\mathfrak{K})\mathfrak{L} = \mathfrak{L}$, then $e\mathfrak{A}(\mathfrak{K})\mathfrak{A}(G) = \bigcup_{\mathfrak{L} \in \mathfrak{L}} \mathfrak{L}$. Since $\mathfrak{C}(\mathfrak{K})$ may be written as $(e_1) \oplus \cdots \oplus (e_m)$, each $e_i$ a primitive idempotent, then $\mathfrak{A}(G) = e_1\mathfrak{A}(\mathfrak{K})\mathfrak{A}(G) + \cdots + e_m\mathfrak{A}(\mathfrak{K})\mathfrak{A}(G)$ is a direct decomposition of $\mathfrak{A}(G)$ into left ideals. Observing that $e_i - Ge_iG^{-1}$ annihilates $\mathfrak{A}(G)$ from the left, any $G \in \mathfrak{G}$, we conclude that $\mathfrak{C}(\mathfrak{K}) \subseteq \mathfrak{C}(G)$.

Indicative of the inconclusiveness of the modular case is

**Theorem 4.** If $\mathfrak{K}$ is a subgroup of $G$ possessing property $\mathfrak{A}$ over the field $\mathfrak{F}$ of characteristic $p$ and if all the irreducible representations of $\mathfrak{K}$ over $\mathfrak{F}$ are one dimensional, then $G$ is an extension of a $p$-group by an abelian group of order $q$, $(q, p) = 1$. Conversely, if $G$ is an extension of a $p$-group by an abelian group of order $q$, $(q, p) = 1$, then any subgroup $\mathfrak{K}$ of $G$ possesses property $\mathfrak{A}$ relative to a field of characteristic $p$.

Since an irreducible $\mathfrak{K}$-module has dimension one, property $\mathfrak{A}$ implies that each irreducible $G$-module is one dimensional over $\mathfrak{F}$. Therefore $\mathfrak{A}(G) = \mathfrak{A}(G) - \mathfrak{N}(G)$ is a commutative algebra. If $G' = \mathfrak{C}(G)$ is the commutator subgroup of $G$ and if $\mathfrak{L}$ is the ideal of $\mathfrak{A}(G)$ generated by the differences $G_i - G_j$, all $G_i, G_j \in G'$, then clearly $\mathfrak{L} \subseteq \mathfrak{N}(G)$. This means that $G'$ is a $p$-group (cf. [3]), and the remainder of the theorem is obvious.
Throughout the remainder of the paper the field $\mathcal{F}$ will be assumed to have characteristic 0 or $p$ with $(p, g) = 1$, $g$ the order of $\mathcal{F}$. Then the next result completely characterizes property $\mathcal{S}$ over $\mathcal{F}$.

**Theorem 5.** Let $\mathcal{K}$ be a normal subgroup of $\mathcal{G}$ of order $h$ and let $\mathcal{K}$ contain $s$ $\mathcal{K}$-conjugate classes. Then $\mathcal{K}$ possesses property $\mathcal{S}$ over $\mathcal{F}$ if and only if $\mathcal{G}$ contains $ns$ $\mathcal{G}$-conjugate classes, where $g = hn$.

Let $e$ be a primitive idempotent from the center of $\mathfrak{A}(\mathcal{K})$. Then $\mathcal{I} = e\mathfrak{A}(\mathcal{K})$ is a minimal two-sided ideal of $\mathfrak{A}(\mathcal{K})$ of order $t^2$. If $\mathcal{K}$ possesses property $\mathcal{S}$, then by Theorem 3 $e$ is a central idempotent of $\mathfrak{A}(\mathcal{G})$ and therefore $\mathcal{B} = e\mathfrak{A}(\mathcal{K})\mathfrak{A}(\mathcal{G})$ is a two-sided ideal of $\mathfrak{A}(\mathcal{G})$ of order $nt^2$. Since $\mathcal{I}$ is orthogonal with $\mathfrak{A}(\mathcal{K}) - \mathcal{I}$ it follows that each minimal $\mathcal{K}$-submodule of $\mathcal{B}$ is isomorphic with a minimal $\mathcal{K}$-submodule of $\mathcal{I}$ and hence is of order $t$. Then it follows from property $\mathcal{S}$ that each minimal left or right ideal of $\mathcal{B}$ is of order $t$, and therefore $\mathcal{B}$ is expressible as a direct sum of $n$ two-sided ideals of $\mathfrak{A}(\mathcal{G})$, each of order $t^2$. Since the dimension of the center of $\mathfrak{A}(\mathcal{K})$ is $s$ this implies that $\mathfrak{A}(\mathcal{G})$ decomposes into a direct sum of $ns$ minimal ideals. Hence $\mathcal{G}$ contains $ns$ conjugate classes.

Conversely, suppose $\mathcal{G}$ possesses $ns$ conjugate classes. Since $\mathcal{K}$ has $s$ conjugate classes, $\mathfrak{A}(\mathcal{K}) = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_s$ and this decomposition is unique. Now if $G \in \mathcal{G}$, $A \in \mathfrak{A}(\mathcal{K})$, the mapping $\theta_G: A \mapsto A^G = GAG^{-1}$ is an automorphism of $\mathfrak{A}(\mathcal{K})$ and $\mathcal{I}_j^G$ is a minimal ideal $\mathcal{I}_j$ of $\mathfrak{A}(\mathcal{K})$. Therefore, under the set of all automorphisms induced by inner automorphisms of $\mathcal{G}$, the minimal ideals $\mathcal{I}$ of $\mathfrak{A}(\mathcal{K})$ separate into non-overlapping sets of transitivity, $S_1$, $\cdots$, $S_m$. That is, if $S_i$ consists of the ideals $\mathcal{I}_{i,1}, \cdots, \mathcal{I}_{i,d(i)}$, then $\mathcal{I}_j^G = \mathcal{I}_{ik}$, $1 \leq k \leq d(i)$, for any $G \in \mathcal{G}$, and given any pair $\mathcal{I}_{ip}$ and $\mathcal{I}_{iq}$ there exists an element $G$ in $\mathcal{G}$ such that $\mathcal{I}_{iq} = \mathcal{I}_{ip}^G$. Then $\mathcal{B}_j = (\mathcal{I}_{i,1} + \cdots + \mathcal{I}_{i,d(i)})\mathfrak{A}(\mathcal{G})$ is a two-sided ideal of $\mathfrak{A}(\mathcal{G})$ of order $n t_i^2 d(i)$, $t_i^2$ the order of $\mathcal{I}_{ij}$.

Let $\mathfrak{L}$ be a minimal left ideal of $\mathfrak{B}_i$. Then $\mathfrak{I}_{ij}\mathfrak{L} \neq (0)$ for some $j$ and therefore, because of the transitivity of $S_i$, for all $j$. Since $\mathcal{I}_v \mathfrak{L}$ is necessarily of order $\geq t_i$ and since $\mathcal{I}_{ij}\mathfrak{I}_{ij} = \delta_{ij}\mathcal{I}_{ij}$, this implies that the order of $\mathfrak{L}$ is $\geq t_i d(i)$. Therefore a minimal two-sided ideal of $\mathfrak{B}_i$ is of order $\geq t_i^2 [d(i)]^2$, and so no decomposition of $\mathfrak{B}_i$ contains more than $n/d(i)$ two-sided ideals. Therefore $\mathfrak{A}(\mathcal{G})$ decomposes into a sum of not more than $n(1/d(1) + \cdots + 1/d(m))$ minimal ideals. However, since $\mathcal{G}$ contains $ns$ conjugate classes, $\mathfrak{A}(\mathcal{G})$ decomposes into a direct sum of $ns$ minimal ideals. Hence $d(1) = \cdots = d(m) = 1$, $m = s$, and each minimal left ideal $\mathfrak{L}$ of $\mathfrak{B}_i$ is of order $t_i$. Since $\mathfrak{L}$ is a left $\mathcal{I}_i$-module whose order equals the order of a minimal left ideal of $\mathcal{I}_i$ it follows that $\mathcal{K}$ possesses property $\mathcal{S}$.
Berman has proved [1] that if $\mathcal{K}$ is a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{K}$ is cyclic of order $n$ and if each of $\mathcal{G}$-conjugate classes $C_i$ contained in $\mathcal{K}$ splits into $k_i \mathcal{K}$-conjugate classes, then $\mathcal{G}$ contains $n(k_1^{-1} + \cdots + k_s^{-1})$ conjugate classes. This result and the previous theorem yield a partial converse to Theorem 3:

**Theorem 6.** If $\mathcal{G}$ is an extension of $\mathcal{K}$ by a cyclic group and if each conjugate class of $\mathcal{K}$ is also a conjugate class of $\mathcal{G}$, then $\mathcal{K}$ possesses property $\mathcal{G}$ over $\mathcal{G}$.

3. **Group extensions by abelian groups.** Obviously the trivial extension $\mathcal{G}$ of a group $\mathcal{K}$ by an abelian group $\mathcal{Q}$, $\mathcal{G} = \mathcal{K} \times \mathcal{Q}$, contains a normal subgroup $\mathcal{K}' \cong \mathcal{K}$ possessing property $\mathcal{G}$ over $\mathcal{G}$. Is the trivial extension the only one for which this is so? We shall see that the answer to this depends on whether or not the order $c$ of $\mathcal{K}$ is prime to the order $n$ of $\mathcal{G}/\mathcal{K}$.

If $\mathcal{K}$ possesses property $\mathcal{G}$ in $\mathcal{G}$ then we have seen that $\mathcal{K}$ is normal in $\mathcal{G}$ and that $\mathcal{G}$ induces class-preserving automorphisms on $\mathcal{K}$. Then the additional condition, $(c, n) = 1$, permits us to apply a result due to M. Hall [4, Theorem 6.1] and to conclude that $\mathcal{G}$ is a trivial extension of $\mathcal{K}$.

In the other direction we prove the following:

**Lemma.** If $\mathcal{K}$ is a group containing a $q$-subgroup $\mathcal{Q}$, $q$ a prime, in its center, then there exists a nontrivial extension $\mathcal{G}$ of $\mathcal{K}$ such that $\mathcal{G}$ contains a subgroup $\mathcal{K}' \cong \mathcal{K}$ possessing property $\mathcal{G}$ in $\mathcal{G}$, $\mathcal{G}/\mathcal{K}'$ of order $q$.

Let $A$ be a generator of a cyclic $q$-subgroup of $\mathcal{K}$ which is of maximal order $q^r$ among those contained in the center of $\mathcal{K}$. Let $x$ be an indeterminate and define $\mathcal{G}$ to be the set of all ordered pairs $(x^i, H)$ where $0 \leq i < q$, $x^0 = 1$, and $H$ is an element of $\mathcal{K}$. Then multiplication in $\mathcal{G}$ is determined by the following definitions: $(x, H_0)^q = (1, A)$, where $H_0$ is the identity element of $\mathcal{K}$, and $(x^i, H_j)(x^k, H_n) = (x^{m}, A^jH_{i+j}H_n)$ where $i+j = m+q$, $0 \leq m < q$. It is easy to verify that $\mathcal{G}$ is a group containing a subgroup $\mathcal{K}' = (1, \mathcal{K}) \cong \mathcal{K}$ possessing property $\mathcal{G}$ in $\mathcal{G}$. Furthermore $\mathcal{G}$ is not isomorphic with the trivial extension of $\mathcal{K}$ since it contains a cyclic $q$-subgroup of order $q^{r+1}$ in its center.

To summarize these results:

**Theorem 7.** If a subgroup $\mathcal{K}$ of a group $\mathcal{G}$ possesses property $\mathcal{G}$ relative to $\mathcal{G}$ then $\mathcal{G}$ may be a nontrivial extension of $\mathcal{K}$ but only if the order of $\mathcal{G}/\mathcal{K}$ is not prime to the order of $\mathcal{K}$. 

A RING ADMITTING MODULES OF LIMITED DIMENSION

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Let $K$ be a ring with unit. A module\(^1\) $M$ over $K$ is said to be finite dimensional if it (i) is finitely based, and (ii) contains no infinite independent set. For such a module there must exist [1, Theorem 7, p. 245] an integer $n$ such that all bases have length $n$ (the invariant basis number property), and no independent set has length greater than $n$. It was shown in a recent paper [1, Theorem 6, p. 245] that this property carries downward with decreasing length of basis. That is: If $K$ admits a module of finite dimension $n$, then every module over $K$ having a basis of length $\leq n$ is also finite dimensional.

It was remarked (in [1]) that this leaves open the possibility that a ring could exist admitting only modules of limited dimension. That is, for some fixed integer $n$ there might exist a ring $K$ such that a module over $K$ is finite dimensional if and only if it has a basis of length $\leq n$. It is the purpose of this paper to construct such a ring for arbitrary $n$.

Let $R$ be the ring of (noncommutative) polynomials generated over the field of integers modulo 2 by a countably infinite set of symbols $\{x_i, y_j\}$, with $i = 1, \ldots, m = (n+2)(n+1); j = 1, 2, \ldots$, where $n$ is the fixed integer chosen. Let $R'$ be the subring of $R$ generated by the $\{x_i\}$. It is desired to order a (suitably restricted) set of $n$-dimensional row vectors of members of $R'$. Begin by ordering the set of all

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\(^1\) Throughout this paper "module" will mean "left module."