A REMARK CONCERNING A THEOREM OF B. FRIEDMAN

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In [1], the following theorem is stated: let $T$ be a densely defined linear operator with closed range in the Hilbert space $\mathcal{H}$, with a densely defined adjoint $T^*$ also having closed range. Let $\phi, \psi$ be vectors in $\mathcal{H}$, and let $\phi \otimes \psi$ be the operator defined by $\phi \otimes \psi(x) = (x, \phi)\psi$. Then $T + \phi \otimes \psi$ also has closed range.

Of course, the fact that $T^*$ is densely defined implies that $T$ is pre-closed; but an examination of the proof shows that it actually requires that $T$ be a closed operator. Under this assumption, a simpler proof can be given; and the need for some such condition will be shown by example.

**Theorem.** Let $T$ be a closed, densely defined operator with closed range. Then $S = T + \phi \otimes \psi$ also has a closed range.

**Proof.** The nullspace $\mathfrak{N}_T$ of a closed operator $T$ is closed, and its domain $\mathcal{D}_T$ is the sum of the two subspaces $\mathfrak{N}_T$ and $\mathcal{D}_T \cap \mathfrak{N}_T = \mathfrak{S}_T$, since $x$ in $\mathcal{D}_T$ can be written $(x - P\mathfrak{N}_Tx) + P\mathfrak{N}_Tx$ (where $P\mathfrak{N}_T$ denotes the orthogonal projection on the subspace $\mathfrak{N}$). If we use the graph norm on $\mathcal{D}_T$, given by the inner product $\langle x, y \rangle = (x, y) + (Tx, Ty)$, then $\mathfrak{N}_T$ and $\mathfrak{S}_T$ are complete, and $\mathcal{D}_T$ is their Hilbert space direct sum.

$T$ restricted to $\mathfrak{S}_T$ is a 1-1 continuous operator from $\mathfrak{S}_T$ (in the graph norm) to the range $\mathfrak{R}_T$ of $T$. The closed graph theorem tells us that its inverse $R$ is continuous, as an operator from the Hilbert space $\mathfrak{R}_T$ to $\mathfrak{S}_T$. Now, the orthogonal complement $[\mathfrak{R}]^\perp$ of $\phi$ in $\mathcal{H}$ is closed in $\mathcal{H}$. Thus its intersection with $\mathfrak{S}_T$ is closed in the graph norm. Then $R^{-1}([\phi]^\perp \cap \mathfrak{S}_T) = T([\phi]^\perp \cap \mathfrak{S}_T)$ is closed in $\mathcal{H}$, $T([\phi]^\perp \cap \mathfrak{S}_T) = S([\mathfrak{R}]^\perp \cap \mathfrak{S}_T) \subset \mathfrak{R}_S \subset \mathfrak{R}_T + [\psi] = T(\mathfrak{S}_T) + [\psi]$. Now, the codimension of $T([\phi]^\perp \cap \mathfrak{S}_T)$ in $T(\mathfrak{S}_T) + [\psi]$ is at most two, so that of $T([\phi]^\perp \cap \mathfrak{S}_T)$ in $\mathfrak{R}_s$ is again at most two. Since $T([\phi]^\perp \cap \mathfrak{S}_T)$ is closed, $\mathfrak{R}_s$ is also closed.

**Remark 1.** If $T$ had been merely preclosed, but with closed range, then it is easy to see $\mathfrak{R}_T = \mathfrak{R}_T$, so that $\mathfrak{S}_T = \mathfrak{T}_T + \phi \otimes \psi$ has closed range.

**Remark 2.** Here is an example of an operator $T$ which is densely defined, bounded, and has closed range, and whose adjoint $T^*$ is therefore bounded and has closed range, but for which $S = T + \phi \otimes \psi$ will not have closed range, for certain $\phi$ and $\psi$.

Let $\mathcal{H}_0$ be a proper infinite-dimensional subspace of $\mathcal{H}$, and

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ψ, ψ₁, ψ₂, ψ₃, ⋯ an orthonormal basis for ℋ₀. Let ψₙ = \( n^{-1/2}(n-1)^{1/2}\psi + \psi_n \). Then the set \( \Psi = \{\psi, \psi', \psi'', \cdots \} \) is linearly independent, and \( \|\psi_n - \psi\|^2 \to 0 \). Enlarge \( \Psi \) to a maximal linearly independent set \( \Phi \) in ℋ₀. Thus, the linear combinations of elements of \( \Phi \) span ℋ₀. Let \( \mathcal{K}_0 \) be the set of all linear combinations of elements of \( \Phi - \{\psi\} \). Then \( \mathcal{K}_0 \) has the following properties:

1. \( \mathcal{K}_0 \) is dense in ℋ₀.
2. \( \psi \in \mathcal{K}_0 \).
3. \( \mathcal{K}_0 + [\psi] = \mathcal{K}_0 \).

Let \( \phi \) be any unit vector in \( \mathcal{K}_0^⊥ \). Let \( T \) be the restriction of \( P_{\mathcal{K}_0} - \phi \otimes \psi \) to \( \mathcal{K}_0 + \mathcal{K}_0^⊥ \). Then clearly \( \mathcal{K}_0 \subset \mathcal{K}_0 + \mathcal{K}_0^⊥ \). Further, \( T|_{\mathcal{K}_0} = P_{\mathcal{K}_0}|_{\mathcal{K}_0} \), so \( \mathcal{K}_0 \subset \mathcal{K}_0 + \mathcal{K}_0^⊥ \). Finally, \( T\phi = -\psi \), so \( \mathcal{K}_0 + \mathcal{K}_0^⊥ \). Notice also that \( T^* = P_{\mathcal{K}_0} - \psi \otimes \phi \), and so if \( x = x_1 + x_2 + \alpha \psi \), where \( x_1 \in \mathcal{K}_0 \cap [\psi]^⊥ \), \( x_2 \in \mathcal{K}_0^⊥ \), then \( T^*x = x_1 + \alpha(\psi - \phi) \). Thus \( \mathcal{K}_0 = (\mathcal{K}_0 \cap [\psi]^⊥) + [\psi - \phi] \), clearly closed. However, \( T + \phi \otimes \psi = P_{\mathcal{K}_0}|_{\mathcal{K}_0} \) has \( \mathcal{K}_0 \) as its range.

**Reference**


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