A REMARK CONCERNING A THEOREM OF B. FRIEDMAN

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In [1], the following theorem is stated: let $T$ be a densely defined linear operator with closed range in the Hilbert space $\mathcal{H}$, with a densely defined adjoint $T^*$ also having closed range. Let $\phi, \psi$ be vectors in $\mathcal{H}$, and let $\phi \otimes \psi$ be the operator defined by $\phi \otimes \psi(x) = (x, \phi)\psi$. Then $T + \phi \otimes \psi$ also has closed range.

Of course, the fact that $T^*$ is densely defined implies that $T$ is pre-closed; but an examination of the proof shows that it actually requires that $T$ be a closed operator. Under this assumption, a simpler proof can be given; and the need for some such condition will be shown by example.

**Theorem.** Let $T$ be a closed, densely defined operator with closed range. Then $S = T + \phi \otimes \psi$ also has a closed range.

**Proof.** The nullspace $\mathcal{N}_T$ of a closed operator $T$ is closed, and its domain $\mathcal{D}_T$ is the sum of the two subspaces $\mathcal{N}_T$ and $\mathcal{D}_T \cap \mathcal{N}_T = S_T$, since $x$ in $\mathcal{D}_T$ can be written $(x - P_{\mathcal{N}_T}x) + P_{\mathcal{N}_T}x$ (where $P_{\mathcal{N}_T}$ denotes the orthogonal projection on the subspace $\mathcal{N}$). If we use the graph norm on $\mathcal{D}_T$, given by the inner product $\langle x, y \rangle = (x, y) + (Tx, Ty)$, then $\mathcal{N}_T$ and $S_T$ are complete, and $\mathcal{D}_T$ is their Hilbert space direct sum.

$T$ restricted to $S_T$ is a 1-1 continuous operator from $S_T$ (in the graph norm) to the range $R_T$ of $T$. The closed graph theorem tells us that its inverse $R$ is continuous, as an operator from the Hilbert space $\mathcal{N}_T$ to $S_T$. Now, the orthogonal complement $[\phi]_T^\perp$ of $\phi$ in $\mathcal{N}$ is closed in $\mathcal{N}$. Thus its intersection with $S_T$ is closed in the graph norm. Then $R^{-1}([\phi]_T^\perp \cap S_T) = T([\phi]_T^\perp \cap S_T)$ is closed in $\mathcal{N}$. $T([\phi]_T^\perp \cap S_T) = S([\phi]_T^\perp \cap S_T) \subset \mathcal{R}_S \subset \mathcal{N}_T + [\psi] = T(S_T) + [\psi]$. Now, the codimension of $T([\phi]_T^\perp \cap S_T)$ in $T(S_T) + [\psi]$ is at most two, so that of $T([\phi]_T^\perp \cap S_T)$ in $\mathcal{R}_S$ is again at most two. Since $T([\phi]_T^\perp \cap S_T)$ is closed, $\mathcal{R}_S$ is also closed.

**Remark 1.** If $T$ had been merely preclosed, but with closed range, then it is easy to see $\mathcal{R}_T = S_T$, so that $S = T + \phi \otimes \psi$ has closed range.

**Remark 2.** Here is an example of an operator $T$ which is densely defined, bounded, and has closed range, and whose adjoint $T^*$ is therefore bounded and has closed range, but for which $S = T + \phi \otimes \psi$ will not have closed range, for certain $\phi$ and $\psi$.

Let $\mathcal{H}_0$ be a proper infinite-dimensional subspace of $\mathcal{H}$, and

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551
Let \( \psi, \psi_1, \psi_2, \psi_3, \cdots \) an orthonormal basis for \( \mathcal{K}_0 \). Let \( \psi_n = n^{-1/2}((n-1)^{1/2} \psi + \psi_n) \). Then the set \( \Psi = \{ \psi, \psi_1, \psi_2, \cdots \} \) is linearly independent, and \( \| \psi_n - \psi \|_2 \rightarrow 0 \). Enlarge \( \Psi \) to a maximal linearly independent set \( \Phi \) in \( \mathcal{K}_0 \). Thus, the linear combinations of elements of \( \Phi \) span \( \mathcal{K}_0 \). Let \( \mathcal{K}_0 \) be the set of all linear combinations of elements of \( \Phi - \{ \psi \} \). Then \( \mathcal{K}_0 \) has the following properties:

1. \( \mathcal{K}_0 \) is dense in \( \mathcal{K}_0 \).
2. \( \psi \in \mathcal{K}_0 \).
3. \( \mathcal{K}_0 + [\psi] = \mathcal{K}_0 \).

Let \( \phi \) be any unit vector in \( \mathcal{K}_0^\perp \). Let \( T \) be the restriction of \( P_{\mathcal{K}_0} - \phi \otimes \psi \) to \( \mathcal{K}_0 + \mathcal{K}_0^\perp \). Then clearly \( \mathcal{K}_0 \subset \mathcal{K}_0 \). Further, \( T|_{\mathcal{K}_0} = P_{\mathcal{K}_0}|_{\mathcal{K}_0} \), so \( \mathcal{K}_0 \subset \mathcal{K}_0 \). Finally, \( T\phi = -\psi \), so \( \mathcal{K}_0 = \mathcal{K}_0 \). Notice also that \( T^* = P_{\mathcal{K}_0} - \psi \otimes \phi \), and so if \( x = x_1 + x_2 + \alpha \psi \), where \( x_1 \in \mathcal{K}_0 \cap [\psi]^\perp \), \( x_2 \in \mathcal{K}_0^\perp \), then \( T^*x = x_1 + \alpha(\psi - \phi) \). Thus \( \mathcal{K}_0 \cap [\psi]^\perp + [\psi - \phi] \), clearly closed. However, \( T + \phi \otimes \psi = P_{\mathcal{K}_0}|_{\mathcal{K}_0} \) has \( \mathcal{K}_0 \) as its range.

Reference


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