

THE INTERIOR POINTS OF THE PRODUCT OF TWO SUBSETS OF A LOCALLY COMPACT GROUP

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1. **Introduction.** Recently, B. J. Pettis conjectured that under certain hypotheses (see Theorem 1 below) the product of two subsets of a locally compact topological group should have a nonvoid interior. The purpose of this paper is to prove this statement. We then use this result to prove, among other things, a theorem of Simon's [1], which he established by a different method.

The conjecture is an analog of earlier work by E. J. McShane [2] and B. J. Pettis [3; 4], namely:

THEOREM (MC SHANE). *Let G be a topological group and let $A, B \subset G$ be sets of second category with one of them satisfying the condition of Baire. Then the interior of AB is not empty.*

Our theorem reads:

THEOREM 1. *Let G be a locally compact topological group with completed Haar measure μ and outer measure μ^* . Let $A, B \subset G$ be sets such that $\mu(A) > 0$ and $\mu^*(B) > 0$. Then the interior of BA (also AB) is nonvacuous.*

In order to show this, we prove the following lemma:

LEMMA 1.1. *Let G be a locally compact group with completed Haar measure μ and outer measure μ^* . Let $A, B \subset G$ be sets such that A has finite measure and B is contained in a set of finite measure. If the function f is defined by $f(x) = \mu^*(xA \cap B)$, then f is continuous and $\int_G f(x) d\mu(x) = \mu(A^{-1})\mu^*(B)$.*

As an application of Theorem 1, we prove:

THEOREM 3 (SIMON [1]). *In a compact group, every subsemigroup which contains a set of positive measure is an open and closed subgroup; and therefore is itself measurable.*

2. Proof of Theorem 1. We first prove Lemma 1.1 which was stated in the introduction.

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PROOF OF LEMMA 1.1. To show f is continuous, let $\epsilon > 0$ be given. We can find a neighborhood U of the identity with the property that $\mu(xA\Delta xuA) = \mu(A\Delta uA) < \epsilon$ for all $u \in U$ and $x \in G$. Now $(xA \cap B) \cdot \Delta(xuA \cap B) = (xA\Delta xuA) \cap B$; hence $\mu^*((xA \cap B)\Delta(xuA \cap B)) < \epsilon$, if $u \in U$. Since xA and xuA are measurable, we have

$$\mu^*(xA \cap B) = \mu^*((xA \cap B) \cap xuA) + \mu^*((xA \cap B) \setminus xuA),$$

and

$$\mu^*(xuA \cap B) = \mu^*((xuA \cap B) \cap xA) + \mu^*((xuA \cap B) \setminus xA).$$

Applying these formulas to the definition of f , it follows that $|f(x) - f(xu)| = |\mu^*((xA \cap B) \setminus xuA) - \mu^*((xuA \cap B) \setminus xA)| \leq \mu^*((xA \cap B)\Delta(xuA \cap B)) < \epsilon$ for all $u \in U$ and $x \in G$; and f is continuous.

Now let $\mu(A^{-1}) = \alpha < \infty$ and $\mu^*(B) = \beta < \infty$, and let the sequence of sets $\{B_n\}$ be chosen so that $B \subset B_{n+1} \subset B_n$; B_n is measurable; and $\mu(B_n) \leq \beta + 1/n$, for $n = 1, 2, \dots$. Let $f_n(x) = \mu(xA \cap B_n)$. Then $f_n \geq f_{n+1} \geq 0$; f_n is continuous; and $\int f_n(x) d\mu(x) = \int \mu(xA \cap B_n) d\mu(x) = \int (\int \chi_{xA}(y) \chi_{B_n}(y) dy) dx = \int \chi_{B_n}(y) (\int \chi_{xA^{-1}}(x) dx) dy = \mu(A^{-1})\mu(B_n)$ for $n = 1, 2, \dots$. Now $\alpha\beta = \mu(A^{-1})\mu^*(B) \leq \mu(A^{-1})\mu(B_n) \leq \alpha(\beta + 1/n)$, for all n . Therefore, if we set $\bar{f} = \inf_n f_n$, then \bar{f} is measurable and $\int_G \bar{f}(x) d\mu(x) = \alpha\beta$. It remains only to show $f(x) = \bar{f}(x)$, i.e. that

$$\mu^*(xA \cap B) = \inf_n \mu(xA \cap B_n).$$

We know that

$$\begin{aligned} \mu^*(B) &= \mu^*(xA \cap B) + \mu^*(B \setminus xA); \\ \mu(B_n) &= \mu(xA \cap B_n) + \mu(B_n \setminus xA); \end{aligned}$$

and clearly,

$$\mu(xA \cap B_n) \geq \mu^*(xA \cap B) \text{ and } \mu(B_n \setminus xA) \geq \mu^*(B \setminus xA).$$

Therefore, since $\mu^*(B) = \inf_n \mu(B_n)$, it must be that $\mu^*(xA \cap B) = \inf_n \mu(xA \cap B_n)$ and the lemma is proved.

REMARKS. (1) Note that we have actually proved f is uniformly continuous. (2) Easy examples, such as a strip in the plane, show that the assertion is false when the measure of A is infinite.

PROOF OF THEOREM 1. There is no loss of generality in assuming A and B have finite outer measure. Define $f(x) = \mu^*(xA^{-1} \cap B)$, and let $V = \{x: f(x) > 0\}$; by Lemma 1.1, V is a nonempty open set. If $x \in V$, then $xA^{-1} \cap B$ cannot be empty, i.e., there are elements a, b

of A and B , respectively, such that $xa^{-1}=b$; $x=ba$. This argument shows that $V \subset BA$ and concludes the proof.

We remark here that, in general, the above theorems are not true if neither A nor B is required to be measurable.

3. Applications. Obviously, Theorem 1 provides us with considerable information concerning semigroups in a topological group. We formulate, here, some of the applications. Unless otherwise stated, G is assumed to be a locally compact topological group with invariant Haar measure μ .

THEOREM 2. *Let S be a subsemigroup of G which contains a subset of positive measure. Then the interior of S is not void. In particular, (1) if S is a group then S is open and closed, and (2) if G is n -dimensional Euclidean space E^n , then S contains an open sector.*

PROOF. The first assertion is a consequence of Theorem 1 and the definition of a semigroup. If a group contains a nonempty open set, then it is open and closed which implies (1), while (2) follows from the observation that every open set in E^n generates a semigroup containing a sector.

PROOF OF THEOREM 3. (The statement appears in the introduction.) By Theorem 2, if S is a semigroup which contains a set of positive measure, then S^0 (= the interior of S) is nonvoid. Now S^0 is an open semigroup, since the interior of a semigroup is again a semigroup, and we now employ a result of Fred B. Wright [5] which asserts that an open semigroup of a compact group is a closed subgroup. Thus, the identity e of G is in the interior of S , and it follows that S is open; S is an open and closed subgroup.

We have shown that for "large" A and B it often happens that AB contains an open set. On the other hand if G is a commutative compact group with an element of infinite order, it will follow from Theorem 3 that there are subsets S and T such that $S \cup T = G$ and neither S^2 nor T^2 contains a nonvoid open set.

To prove this last statement, we will need the following purely algebraic lemma:

LEMMA 4.1. *Let H be an Abelian group and let $h \in H$. If $S \subset H$ is a semigroup containing h but not containing $-h$, and if S is maximal with respect to this property, then $S \cup -S = H$.*

PROOF. Suppose, to the contrary, that there is an element $t \in H$ such that $t \notin S \cup -S$. Obviously $-t \notin S \cup -S$, so by the maximality of S , it follows that there is some positive integer n and $s_1 \in S$ with

$-h = s_1 + n(-t)$. Therefore, we may now pick n' to be the smallest positive integer with the property $n't \in S$. Similarly, there is a positive integer m and an $s_2 \in S$ such that $-h = s_2 + mt$. There exist positive integers p, r with $0 \leq r < n'$ and $m = pn' - r$; but $rt = -mt + pn't = s_2 + h + pn't \in S$; hence $r = 0$. This gives $-h = s_2 + pn't \in S$, a contradiction, and the proof is complete.

As for the statement preceding Lemma 4.1, assume that G is a compact Abelian group having an element of infinite order. By Zorn's Lemma, the S of Lemma 4.1 exists. Moreover, $S^2 \subset S$ cannot contain an open set or S would be a group, by Theorem 3; the same statement holds for $-S$; and $S \cup -S = G$.

We now make the further claim that S is not measurable. For, if it were, then its measure must be positive and its interior would be nonempty. In particular, we have proved the following:

THEOREM 4. *In a compact Abelian group, every maximal semigroup which is not a group is not Haar measurable.*

Of course, such semigroups may not exist, but if the group has one element of infinite order, then it can be shown that there is a maximal semigroup which is not a group.

Our final theorem shows that the term "semigroup" is not a misnomer. Roughly, it says that if a semigroup is "more than one-half of a group," then it is indeed a group.

THEOREM 5. *Let S and T be measurable subsets of G such that (1) S is a semigroup, (2) T is a group, and (3) $\mu(S\Delta T) < \mu(T)/2$. Then $S = T$.*

We first prove the following lemma.

LEMMA 5.1. *If A and B are measurable subsets of G such that (1) A is a semigroup, (2) B is a group, (3) $B \subset A$, and (4) $\mu(A \setminus B) < \mu(B)$. Then $A = B$.*

PROOF. By way of contradiction, suppose that $x \in A \setminus B$. Then, since A is a semigroup and B is a group, we have $(xB) \subset A \setminus B$. But $\mu(xB) = \mu(B) > \mu(A \setminus B)$, and the lemma follows.

In the theorem, first consider the case where $\mu(T) = \infty$. If A is the set of $s \in S \cap T$ such that $s^{-1} \notin S \cap T$, then $A^{-1} \subset T \setminus S$. Therefore $\mu(S \cap S^{-1} \cap T) = \infty$ and $\mu(T \setminus (S \cap S^{-1})) < \infty$. The lemma then implies that $T = S \cap S^{-1} \cap T$. We now have a group T in S satisfying the hypothesis of the lemma; that is $T = S$.

In case $\mu(T) < \infty$, then T is compact; and $S \cap T$ is a semigroup of positive measure in T . So by Theorem 3 and the lemma, $S \cap T = T$; $S = T$.

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SIMPLE NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. **Introduction.** Nodal algebras were defined by R. D. Schafer [4] and have also been studied by the author [2; 3]. A noncommutative Jordan algebra is an algebra \mathfrak{A} over a field \mathfrak{F} satisfying (1) the flexible law $(xy)x = x(yx)$ and (2) the condition that \mathfrak{A}^+ is a Jordan algebra. That is, \mathfrak{A}^+ satisfies the identity $(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x)$ where we have used the dot to indicate the product of \mathfrak{A}^+ . The algebra \mathfrak{A}^+ is defined to be the same vector space as \mathfrak{A} but with product $x \cdot y = (xy + yx)/2$ where xy and yx are products in \mathfrak{A} . Then \mathfrak{A} is called nodal if it is finite dimensional, if \mathfrak{A} has identity element 1, if \mathfrak{A} can be written as a vector space direct sum $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ where \mathfrak{N} is the subspace of nilpotent elements of \mathfrak{A} , and if \mathfrak{N} is not a subalgebra of \mathfrak{A} .

Every known nodal algebra \mathfrak{A} has the property that \mathfrak{A}^+ is an associative algebra. The flexible algebras with \mathfrak{A}^+ associative have been described in [3]. In this paper we shall prove the following theorem.

THEOREM 1. *Let \mathfrak{A} be a simple nodal noncommutative Jordan algebra of characteristic $\neq 2$. Then \mathfrak{A}^+ is associative.*

Define \mathfrak{B} to be the subspace of \mathfrak{A} generated by all the associators in \mathfrak{A}^+ . That is, \mathfrak{B} is generated by elements of the form $(x \cdot y) \cdot z - x \cdot (y \cdot z)$ with x, y, z in \mathfrak{N} . The proof of the theorem will be made by showing that the ideal \mathfrak{C} of \mathfrak{A} generated by \mathfrak{B} is not all of \mathfrak{A} and since \mathfrak{A} is simple it will follow that $\mathfrak{C} = 0$ and $\mathfrak{B} = 0$. This is the desired result.

The original proof was not valid when the characteristic is 3. The author thanks Professor R. D. Schafer for suggesting a modification

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