AN EXTENSION OF A THEOREM OF MANDELBROJT

R. A. KUNZE

1. Introduction. In a recent paper [3] S. Mandelbrojt proved several interesting theorems concerning Fourier transforms and analytic functions. One of these results can be formulated as follows:

(1.1) Suppose $F \in L_\alpha$, $a \geq 0$, and that $k$ is never zero outside the closed interval $[-a, \infty]$ where $k$ is the Fourier transform of a function $K$ in $L_1$ such that $K * F \equiv 0$. Then there exists a function $F_0$ analytic in the right half plane with the properties

(i) $|F_0(x + iy)| \leq \|F\| e^{\alpha x}, \quad x > 0,$

(ii) $\lim_{x \to +0} \int_{-N}^{N} |F_0(x + iy) - F(y)| \, dy = 0.$

It is not difficult to show that the conclusion is satisfied if we assume only that there exists for each $t$ in $(-\infty, -a)$ a function $K$ in $L_1$ (depending upon $t$) such that $K * F \equiv 0$ and $k(t) \neq 0$. Our principal aim in this paper is to extend this latter improved version to $n$-dimensions. That such an extension exists follows almost immediately from a theorem of Y. Fourés and I. E. Segal [1] concerning causal operators and analytic functions (once it is established that a certain bounded operator $T$ determined by $F$ is causal). On the other hand as indicated by these authors some of the results in [1], specifically those pertaining to domains of dependence, admit improvement when treated from the point of view of Banach algebras; as one is led naturally to making these improvements in the course of showing that $T$ is causal we shall begin our discussion at this point.

2. Domains of dependence. Throughout this part $\mathcal{G}$ will denote an arbitrary locally compact abelian group. As is well known, the Plancherel transform $U(U : L_2(\mathcal{G}) \to L_2(\mathcal{G}))$ establishes a one to one correspondence between bounded operators $T$ on $L_2(\mathcal{G})$ that commute with translations and bounded measurable functions $F$ on the dual group $\hat{\mathcal{G}}$. More precisely $T \leftrightarrow F$ if and only if $UTU^{-1} = M_F$ where $M_F$ denotes the operation of multiplication by $F$ on $L_2(\hat{\mathcal{G}})$. We recall that the spectrum or spectral set $\Lambda(F)$ of $F$ is defined as the set of all $x$ in $\mathcal{G}$ such that $k(x) = 0$ whenever $k$ is the (inverse) Fourier transform.

Received by the editors October 15, 1957.

1 This research was supported in whole by the United States Air Force under Contract No. AF49(638)-42, monitored by the AF Office of Scientific Research of the Air Force office.

553
of a function $K$ in $L_2(\mathcal{G})$ such that $K * F \equiv 0$. The set of all such $K$ forms a closed ideal $I$ in $L_2(\mathcal{G})$. A theorem of Segal's [2] asserts that

(2.1) A sufficient condition that $K \in I$ is that its Fourier transform $k$ vanishes outside a compact subset of the complement $\Lambda(F)'$ of $\Lambda(F)$.

**Definition 2.1.** A bounded operator $T$ is said to be dependent upon a subset $E$ of $\mathcal{G}$ if $Tg$ vanishes outside $N + E$ whenever $N$ is compact, $g \in L_2$ and $g$ vanishes outside $N$.

**Definition 2.2.** A bounded operator $T$ has a domain of dependence if there exists a closed subset $E$ of $\mathcal{G}$ such that $T$ is dependent upon $E$ and is not dependent upon any proper closed subset of $E$. $E$ is called a domain of dependence of $T$.

**Theorem 2.1.** Suppose $T$ is a bounded operator on $L_2(\mathcal{G})$ that commutes with translations. Then $T$ has a unique domain of dependence, and if $UTU^{-1} = M_F$, $E$ is precisely the spectrum of $F$.2

**Lemma 2.1.** If $T$ is dependent upon a closed set $E$ then $\Lambda(F) \subseteq E$.

Let $e \in E'$ and choose a compact neighborhood $N$ of 0 such that $E + N$ and $e + N$ are disjoint. Suppose $g, h \in L_2$ and vanish outside $N$, $e + N$. Then $(Tg)h = 0$ and taking Fourier transforms we get $(FG) * H \equiv 0$. Let $x$ be a fixed character of $\mathcal{G}$ (i.e. an element of $\mathcal{G}$) and denote the value of $x$ at $t \in \mathcal{G}$ by $(x \cdot t)$. Replacing $g(t)$ by $k_2(t) = (x \cdot t)[g(t)]^{-1}$ we see that $K(u) = \int (u \cdot t)k_2(t)dt = [G(x - u)]^{-1}$. Hence $\int F(u - v)[G(x - (u - v))]^{-1}H(v)dv = 0$ for all $u$ and putting $u = x$ we get $\int F(x - v)[G(v)]^{-1}H(v)dv = 0$. As $x$ was arbitrary in $\mathcal{G}$ it follows that $F \ast \bar{G}H \equiv 0$. Now by choosing $g$ and $h$ suitably (subject to the above restrictions) we can insure that $g^* \ast h(e) \neq 0$. Since $g^* \ast h$ is the (inverse) Fourier transform of $\bar{G}H$ it follows that $e \in \Lambda(F)'$, and consequently $\Lambda(F) \subseteq E$.

**Proof of the theorem.** Since $\Lambda(F)$ is closed and in view of the result just established it suffices to show that $T$ is dependent upon $E = \Lambda(F)$. Suppose then that $g \in L_2$ and vanishes outside a compact subset $N$. To show that $Tg$ vanishes outside $E + N$ it suffices to show that $(Tg, h) = 0$ for every $h \in L_2$, vanishing outside a compact subset $C$ of $(E + N)'$. By Parseval's formula, it suffices to show that $\int FGH = 0$. Now $\int FGH = \int F[\bar{G}H]^{-1} = F \ast (\bar{G}H) \ast (o)$ and it is therefore sufficient to show that $F \ast (\bar{G}H) \ast \equiv 0$. The Fourier transform of $(\bar{G}H) \ast$ is $[g^* \ast h]^{-1}$ and as $g$ vanishes outside $N$, $g^* \ast h$ vanishes outside $-N$. Thus $[g^* \ast h]^{-1}$ vanishes outside $C - N$. Furthermore $C - N$ is compact and disjoint from $E$. Consequently (2.1) applies and we see that $F \ast (\bar{G}H) \ast \equiv 0$.

Remark. In the case of a real (finite dimensional) vector group the domain of dependence for $T$ in the sense of Fourés-Segal is simply the closed convex set generated by $\Lambda(F)$.

3. Mandelbrojt's theorem. Throughout this part $\mathcal{G}$ will denote $n$-dimensional real Euclidean space regarded as a vector group. The dual of a cone $C$ in $\mathcal{G}$ is the cone $\mathcal{C}$ in the dual group $\hat{\mathcal{G}}$ consisting of all $x$ such that $(x \cdot t) \geq 0$ for all $t$ in $C$. The tube $\Gamma$ over $\mathcal{C}$ is the set of all complex vectors $x + iy$, $x \in \mathcal{C}$, $y \in \hat{\mathcal{G}}$. Putting $v$ for the vertex of $C$ we define the spine of $\Gamma$ to be the subset of all $v + iy$, $y \in \hat{\mathcal{G}}$. By means of an obvious correspondence we can identify functions on $\hat{\mathcal{G}}$ with functions on the spine of $\Gamma$. A function $F_0$ defined on the interior $\Gamma^0$ of $\Gamma$ is said to extend a function $F$ on the spine, or to have boundary values on the spine if any sequence $x_n \to v$ with $x_n$ in $\mathcal{C}^0$ has a subsequence $x_m$ such that $F_0(x_m + iy) \to F(v + iy)$ a.e. relative to Lebesgue measure.

A bounded operator $T$ on $L^2(\mathcal{G})$ is said to be causal with respect to $C$ if $Tg$ vanishes outside $a + C$ whenever $g$ is in $L^2$ and vanishes outside $a + C$, $a$ being arbitrary in $\mathcal{G}$. We shall use the following reformulation of the basic result concerning bounded causal operators given in [1].

(3.1) Suppose $C$ is a closed convex cone with vertex at 0 and nonempty interior. Let $T$ be a bounded operator on $L^2(\mathcal{G})$ that commutes with translations, and suppose $F$ is the unique (modulo null functions) bounded measurable function on $\hat{\mathcal{G}}$ such that $UTU^{-1} = M_F$ where $U$ is the Plancherel transform,

$$U: L^2(\mathcal{G}) \to L^2(\hat{\mathcal{G}})$$

and $M_F$ is the operation of multiplication by $F$ on $L^2(\mathcal{G})$. Then $T$ is causal with respect to $C$ if and only if there exists a function $F_0$ analytic on the interior of the tube $\Gamma$ over the dual of $C$ that extends $F$ and satisfies the additional conditions,

(i) $|F_0(z)| \leq \|F\|_\infty, \quad z \in \Gamma^0$

(ii) $\lim_{z \to 0} \int_D |F_0(x + iy) - F(y)|^2dy = 0$

where $x \to 0$ in $\mathcal{C}^0$ and $D$ is an arbitrary compact subset of the spine of $\Gamma$.

Our extension of Mandelbrojt's theorem reads as follows.

**Theorem 3.1.** Suppose $F \in L^\infty(\hat{\mathcal{G}})$ and that $C$ is a closed convex cone in $\mathcal{G}$ with vertex at 0 and nonempty interior. Let $a$ be a fixed element in $C$. Suppose further that there exists for each $t$ outside $C - a$ a function $K$ in $L^1(\mathcal{G})$ such that $K * F \equiv 0$ and $k(t) \neq 0$ where $k$ is the (inverse) Fourier
transform of \( K \). Then there exists a function \( F_0 \) analytic on the interior of the tube \( \Gamma \) over the dual of \( C \) extending \( F \) and having the additional properties,

(i) \[ |F_0(x + iy)| \leq \|F\|\infty e^{a \cdot z} \]

for all \( x \) in \( \mathcal{C}_0 \) and \( y \) in \( \hat{G} \).

(ii) \[ \lim_{x \to 0} \int_D |F_0(x + iy) - F(y)|^2 dy = 0 \]

where \( x \to 0 \) in \( \mathcal{C}_0 \) and \( D \) is an arbitrary compact subset of the spine of \( \Gamma \).

It is easy to see that the theorem follows, by translation, from the case \( a = 0 \). The details of this reduction are given in the following

**Lemma 3.1.** If the theorem is true for \( a = 0 \) it is true in general.

Suppose \( F \in L_{\infty}(\mathcal{G}) \) and that the additional hypotheses of the theorem are satisfied. Put \( H(x) = e^{-i(a \cdot z)}F(x) \) and suppose \( t \in \mathcal{C}' \). Then \( t - a \in (C - a)' \) and there exists \( K \) in \( L_1(\mathcal{G}) \) such that \( K \ast F = 0 \) and \( k(t - a) \neq 0 \) where \( k(u) = (2\pi)^{-n/2}\int e^{i(u \cdot z)}K(x)dx \). Putting \( L(x) = e^{-i(a \cdot z)}K(x) \) it follows that \( 1(t) = k(t - a) \neq 0 \) and that \( L \ast H(x) = e^{-i(a \cdot z)}K \ast F(x) = 0 \). Assume the theorem is true for the case \( a = 0 \), and let \( H_0 \) be the extension of \( H \). Define \( F_0 \) by

\[ F_0(x + iy) = e^{a \cdot (x + iy)}H_0(x + iy) \]

for \( x \in \mathcal{C}_0 \) and \( y \in \hat{G} \). Then \[ |F_0(x + iy)| \leq e^{a \cdot z}\|H\|\infty = e^{a \cdot z}\|F\|\infty, \]

and if \( D \) is a compact subset of the spine of \( \Gamma \) we have,

\[ \int_D |F_0(x + iy) - F(y)|^2 dy \leq \int_D |e^{a \cdot (x + iy)}H_0(x + iy) - e^{ia \cdot y}H(y)|^2 dy \]

Now \[ \int_D |e^{a \cdot z}H_0(x + iy) - H_0(x + iy)|^2 dy = (e^{a \cdot z} - 1)^2 \int_D |H_0(x + iy)|^2 dy \]

\[ \to 0 \] as \( x \to 0 \) in \( \mathcal{C}_0 \), in view of (ii), and the fact that \( (e^{a \cdot z} - 1)^2 \to 0 \). Combining these estimates we see that \( \int_D |F_0(x + iy) - F(y)|^2 dy \to 0 \) as \( x \to 0 \) through values in \( \mathcal{C}_0 \).

**Proof of the Theorem.** By the preceding lemma we can assume \( a = 0 \). Now let \( T \) be the bounded operator on \( L_2(\mathcal{G}) \) given by the equation \( T = U^{-1}M_FU \). It is apparent from (3.1) that it suffices to show that \( T \) is causal with respect to the cone \( C \). Since \( T \) commutes with translations we need only show that \( Tg \) vanishes outside \( C \) whenever \( g \) is in \( L_2 \) and vanishes outside \( C \). Furthermore as \( T \) is
bounded it suffices, by continuity, to consider the case that $g$ vanishes outside a compact subset $N$ of $C$. Now our assumptions clearly imply that the spectrum $E$ of $F$ is contained in $C$, and, by Theorem 2.1, $T$ is therefore dependent upon $E$. Hence $Tg$ vanishes outside $E+N$. Finally since $C$ is closed under addition, $E+N$ is contained in $C$ which concludes the proof.

References


Massachusetts Institute of Technology