ORDER AND COMMUTATIVITY IN BANACH ALGEBRAS

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S. Sherman has shown [4] that if the self adjoint elements of a C* algebra form a lattice under their natural ordering the algebra is necessarily commutative. In this note we extend this result to real Banach algebras with an identity and arbitrary Banach * algebras with an identity. The central fact for a real Banach algebra $A$ is that if the positive cone is defined to be the uniform closure of the set of finite sums of squares of elements of $A$, and if $A$ is a lattice under the ordering induced by this cone, then extreme points of the unit sphere of the dual cone are multiplicative linear functionals. A similar situation holds for * algebras.

1. Real Banach algebras. Let $X$ be a real Banach space, and let $C$ be a closed cone in $X$. For $x, y \in X$ we define $x \geq y$ if $x - y \in C$. If in addition $X$ is a lattice under the ordering $\geq$, we say $C$ lattice-orders $X$. Let $C'$ be the dual cone and let $\sum = \{f \in C': \|f\| \leq 1\}$. The set of extreme points of $\sum$ will be denoted by $S$. For a real linear functional $f$ let $I_f = \{x \in X: f(x) = 0\}$ and let $R = \bigcap_{f \in C'} I_f$. Lastly if $X$ is a lattice we define $x_+ = x \vee 0$, $x_- = x \wedge 0$, and $|x| = x_+ - x_-$. We note $|x| \geq 0$.

**Lemma 1.** If $C$ is a closed cone in a real Banach space $X$, then

1. $R = C \cap -C$,
2. $R = \bigcap_{f \in S} I_f$,
3. If $C$ lattice-orders $X$, $R = \{0\}$.

**Proof.** Obviously $C \cap -C \subseteq R$. For the converse, by the Hahn-Banach theorem $x \in C$ iff $f(x) \geq 0$ for each $f \in C'$. Therefore $R \subseteq C \cap -C$. For (ii) suppose $x \in \bigcap_{f \in S} I_f$, $x \in R$, then there exists an $f \in C'$, $\|f\| = 1$, such that $|f(x)| = 2\epsilon \neq 0$. But by the Krein-Milman theorem there exist finitely many $f_i \in S$ and real numbers $\alpha_i$ such that $|f(x) - \sum \alpha_i f_i(x)| < \epsilon$. Hence for some $i, f_i(x) \neq 0$, which is a contradiction. Lastly let $C$ lattice order $X$. Then $x \in C$ implies $x \geq 0$ or $x_- = 0$, and $x \in -C$ implies $-x \geq 0$ or $x_+ = 0$. Since $x = x_+ + x_-$, $x \in C \cap -C$ implies $x = 0$.

The central tool in both this investigation and that of Sherman is the following result of Krein and Krein [3]. It can be stated in a

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1 We refer the reader to [2] for the appropriate definitions of cone, dual cone, etc.

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slightly more general fashion, but the following is sufficient for our purposes.

**Theorem 1.** Let $X$ be a real Banach space which is lattice ordered by a cone $C$. Suppose in addition that $C$ contains an element $e$, $\|e\| = 1$, such that $\{y : \|e-y\| \leq 1\} \subseteq C$. Then $f \in S$ iff $|f(x)| = f(|x|)$ for each $x \in X$.

Let us specialize to a real Banach algebra $A$ with identity such that $\|1\| = 1$, and let $C$ be the closure of the set of finite sums of squares of elements of $A$. By the familiar binomial series argument (c.f. [2]), $\{y : \|1-y\| \leq 1\} \subseteq C$. Also, for $f \in C'$ and $x, y \in A$ we have the Schwartz inequality, $|f(xy+yx)|^2 \leq 4f(x^2)f(y^2)$, which may be verified by the classical argument. Also useful is the following property of functionals in $C'$.

**Lemma 2.** Let $A$ be a real Banach algebra, and let $x \in C, f \in C'$. Then $f(x) = 0$ implies $f(x^2) = 0$.

**Proof.** First assume $x = y^2$, and let $f \in C'$. If $\|x\| \leq 1$, the binomial series for $(1-x)^{1/2}$ converges absolutely. Therefore $1-x \in C$. Moreover since $x = y^2$, $y(1-x)^{1/2} = (1-x)^{1/2}y$ and $x(1-x) = [y(1-x)^{1/2}]^2 \in C$. Therefore $x - x^2 \geq 0$. Hence $f(x) = 0$ implies $f(x^2) = 0$. We proceed now by induction. Let $x = \sum_{i=1}^{n+1} y_i^2$ and let $f(x) = 0$. If $z = \sum_{i=1}^{n} y_i^2$, then $f(y_{n+1}^2) = f(z) = 0$. We assume $f(z^2) = 0$, and by the above argument $f(y_{n+1}^4) = 0$. An application of the Schwartz inequality gives us

$$0 \leq f(x^2) = f((z + y_{n+1}^2)^2) = f(zy_{n+1}^2 + y_{n+1}^2z) \leq 2[f(z^2)f(y_{n+1}^4)]^{1/2} = 0.$$ 

Therefore the result holds for all finite sums of squares, and by continuity it holds for all $x$ in $C$.

**Theorem 2.** If $C$ lattice-order $A$ then each $f \in S$ is a homomorphism of $A$ onto the real numbers, and $A$ is commutative.

**Proof.** For $x, y \in A$ define the Jordan multiplication $x \circ y = (xy + yx)/2$. Thus $A$ can be considered as a Jordan ring with an identity. We assert that for $f \in S$, $I_f$ is a Jordan ideal. By Theorem 1 $x \in I_f$ iff $x_+ - x_ \in I_f$. Therefore let $x \geq 0, x \in I_f$. By the Schwartz inequality and Lemma 2, for any $y \in A$, $|f(xy + yx)|^2 \leq 4f(x^2)f(y^2) = 0$. Hence $xy + yx \in I_f$, and since $I_f$ is obviously closed under addition, $I_f$ is a Jordan ideal.

Now a linear functional of any algebra over a field which takes the identity of the algebra into the identity of the field is a homomorphism if its kernel is a two-sided ideal. Hence $f$ is a Jordan homo-
morphism of $A$ onto the reals. On the other hand Jacobson and Rickart [1, Theorem 2] have proved that a Jordan homomorphism of a ring into an integral domain is either a homomorphism or an anti-homomorphism. An application of this result proves that $f$ is a homomorphism.

Finally for each $f \in S$ and $x, y \in A$, $xy - yx \in I_f$. Since by Lemma 1 $\cap_{f \in S} I_f = \{0\}$, $A$ must be commutative.

2. Banach * algebras. Let $A$ be a Banach * algebra with a continuous involution and an identity. Let $C$ be the closure of the set of finite sums of elements $xx^*$. $C$ is a closed cone in the real linear space $H$ of self adjoint elements of $A$. The dual cone of $C$ (in the conjugate space of $H$) can be identified with the set of those functionals $f$ on $A$ for which $f(xx^*) \geq 0$ for each $x \in A$ (cf. [2] for details). Let $\sum S$ be as before and for $f \in C'$ let $I_f = \{x \in A : f(x) = 0\}$ and $R = \cap_{f \in C'} I_f$.

We also note that $\{h \in H : \|1 - h\| \leq 1\} \subset C$ and for $f \in C'$ the familiar Schwartz inequality holds, i.e. $|f(xy^*)|^2 \leq f(xx^*)f(yy^*)$, $x, y \in A$.

**Lemma 3.**

$$R = C \cap -C + i(C \cap -C),$$

$$R = \cap_{f \in S} I_f.$$

**Proof.** Let $T = (C \cap -C) + i(C \cap -C)$. Obviously $T \subset R$. If $x \in R$, let $h = (x + x^*)/2$, $k = (x - x^*)/2i$. Then $h, k$ are self adjoint, $h, k \in I_f$ and $x = h + ik$. But a self adjoint element $y \in C$ iff $f(y) \geq 0$ for each $f \in C'$. Therefore $h, k \in C \cap -C$ and $T = R$. For the second assertion if $x \in \cap_{f \in S} I_f$ and $x \in R$, we may assume $x$ is self adjoint and apply the argument of Lemma 1.

**Lemma 4.** Let $h \in C$ and $f \in C'$. Then $f(h) = 0$ implies $f(h^2) = 0$.

**Proof.** If $h = h^*$, and $\|h\| \leq 1$, then by the familiar series argument $1 - h = k^2$, where $k = k^*$. Therefore $1 - h \in C$. Since $kh = hk$ and $kk \in C$, $kk = kh^2 = h - h^2 \in C$. Therefore $f(h) = 0$ implies $f(h^2) = 0$.

**Theorem 3.** If $C$ lattice-orders $H$, then each $f \in S$ is a homomorphism of $A$ onto the complex numbers, and $A$ is commutative.

**Proof.** To prove that $f$ is a homomorphism it suffices to show that for $f \in S$, $I_f$ is a two-sided ideal. Let $x \in I_f$, $y \in A$. We assert $xy \in I_f$. First we may assume $x$ is self adjoint and by Theorem 1 we may assume $x \geq 0$. But then applying the Schwartz inequality

$$|f(xy)|^2 \leq f(x^2)f(y^*y) = 0.$$
This proves $xy \in A$ and similarly $yx \in A$. Since $I_T$ is obviously closed under addition, it is a two sided ideal. An application of Lemma 3 proves that $A$ is commutative.

**Bibliography**


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**A NOTE ON VALUED LINEAR SPACES**

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Banaschewski [1] has given a simple and elegant proof of Hahn's embedding theorem for ordered abelian groups. His method can be used to prove the author's generalization of Hahn's theorem [2, p. 11]. In this note we make use of Banaschewski's method to prove a special case of the author's theorem (which is also a generalization of Hahn's theorem) that has been proven by Gravett [3].

Let $(L, \Delta, d)$ be a valued linear space [3]. That is, $L$ is a vector space over a division ring $K$, $\Delta$ is a linearly ordered set with minimum element $\theta$, and $d$ is a mapping of $L$ onto $\Delta$ such that for all $x, y \in L$, $d(x) = \theta$ if and only if $x = 0$, $d(x) = d(kx)$ for all $0 \neq k \in K$, and $d(x + y) \leq \text{Max} [d(x), d(y)]$. For each $\delta \in \Delta$, let $C^\delta = \{x \in L : d(x) \leq \delta\}$ and let $C'_\delta = \{x \in L : d(x) < \delta\}$. Let $W$ be the vector space of all mappings $f$ of $\Delta$ into the join of the spaces $C^\delta/C'_\delta$ for which $f(\delta) \in C^\delta/C'_\delta$ and $R_f = \{\delta \in \Delta : f(\delta) \neq C'_\delta\}$ is an inversely well ordered set. $W$ is a subspace of the unrestricted direct sum $V$ of the $C^\delta/C'_\delta$. $W$ is also a valued linear space $(W, \Delta, d')$, with $d'(f)$ the largest $\delta \in R(f)$.

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